ANABELIAN METHODS IN ALGEBRAIC AND ARITHMETIC GEOMETRY

1. Introduction

Much of the richness of algebraic geometry comes from its beautiful interactions with, and in some cases unification of, number theory and topology. The use of these connections was one of the main themes of 20th-century number theory, and many of the great mathematical achievements of that period (for example, the proofs of the Weil conjectures and Fermat's Last Theorem) exploited the interactions between geometry, number theory, and topology.

The proposal under review aims to strengthen the connections between number theory, topology, and algebraic geometry, through the use of new arithmetic and topological techniques in the service of classical algebro-geometric and number-theoretic problems. Over the last few years, I have developed a research program aimed at using recent developments in arithmetic geometry to analyze several diverse problems — for example, the analysis of Galois actions on fundamental groups, the geometric torsion conjecture, the Frey-Mazur conjecture for function fields, and certain aspects of integral p-adic Hodge theory. Broadly speaking, the goal of this program is to use input from number theory to understand the topology of algebraic varieties, and to use input from algebraic geometry and low-dimensional topology to answer classical number-theoretic questions.

Let X be an algebraic variety, i.e. the set of solutions to a system of polynomial equations

$$\begin{cases} f_1(x_1, \cdots, x_m) &= 0 \\ & \vdots \\ f_n(x_1, \cdots, x_m) &= 0. \end{cases}$$

The main observation that lets one use arithmetic techniques to study X is that the polynomials $\{f_i\}$ are defined by a finite set of coefficients S, so one may just as well view the system of equations above as living over the finitely-generated $\mathbb{Z}$-algebra $\mathbb{Z}[S]$. Such systems of equations are accessible to the techniques of arithmetic geometry; for example, one may reduce them modulo a prime, or view them as systems of equations over a p-adic field. In particular, any invariant of X which may be defined *purely algebraically* may be studied via such arithmetic techniques.

The bulk of this proposal is aimed at studying one such invariant via arithmetic methods — namely, the fundamental group. The fundamental group of X has various algebraic incarnations, in profinite, p-adic, de Rham, and other contexts. The main goal of this proposal is:

> To develop the arithmetic theory of fundamental groups of algebraic varieties, by combining explicit ℓ-adic computations, recent developments in integral p-adic Hodge theory, and analytic techniques, and to pursue applications of this theory to:
> (1) the geometric torsion conjecture, the geometric Frey-Mazur conjecture, and related questions about monodromy representations in algebraic geometry.
> (2) the analysis of solutions to Diophantine equations via anabelian methods, and in particular to cases of the section conjecture.

These programs have already met with significant successes (greatly benefited by previous NSF support), described in Section 2. Section 3 describes the anticipated directions of these programs in the coming years. Sections 4 and 5 summarize the broader impacts and intellectual merits of the proposal, respectively.

2. Results from prior NSF Support

Sections 2.2.1, 2.2.2, 2.2.3, 2.2.4 and 2.2.5 describe work supported by the Mathematical Sciences Postdoctoral Research Fellowship (MSPRF, Award No. 1502386, for $150,000), active 9/15/2015-8/31/2018 or by the NSF Graduate Research Fellowship (Application No. 1000104696). Resulting papers: [Lit14, Lit15, Lit16, Lit18b, Lit18a, Lit17, Lit18c, Litar, LLar, LL19, JL19].

2.1. Broader Impacts.

I have used my prior NSF support to broadly disseminate mathematics, both domestically and internationally. In the three years that I was supported by the MSPRF, I gave more than 50 invited talks on my research, at conferences, colloquia, and research seminars. The interdisciplinary nature of my work — involving interactions between arithmetic, geometry, and topology — has fostered several in-progress collaborations, some international. I was also the US organizer for GAeL XXII and XXIII, two international conferences aimed at graduate students and early-career post-docs. In 2017, I ran the study group on p-adic Hodge theory at the Arizona Winter School on perfectoid spaces, which consisted of approximately 40 graduate students. I co-organized the Columbia Algebraic Geometry Seminar from 2016-2018, as well as several other reading seminars.

I have mentored three group undergraduate research projects (jointly with Daniel Halpern-Leistner, David Hansen, and Alex Perry, respectively), and two individual undergraduate research projects. I regularly give expository talks on mathematics to undergraduates and high school students, and exposit mathematics online (both at the undergraduate and graduate level) in the form of a growing mathematical blog, and in a Numberphile video [num19], which as of this writing has more than 300,000 views.

2.2. Intellectual merit.

2.2.1. *Anabelian geometry and monodromy.*

Much of my recent work has focused on Galois actions on étale fundamental groups, and on the representation theory of arithmetic fundamental groups. While I think the most exciting applications of this work are yet to come (see Section 3.1), the theory I have developed has already had significant applications to the study of monodromy representations. In principle, this study is motivated by the following question:

Question 2.2.1. Let K be a field. Can one classify all K-varieties, and all morphisms between them?

As stated, this question is ill-posed, and it is hard to imagine a well-posed version of it. However, we may linearize Question 2.2.1. Namely, if X is a K-variety, the Galois group of K acts on the ℓ-adic cohomology of $X_{\bar{K}}$. That is, for each $i \geq 0$, we obtain a representation

$$\mathrm{Gal}(\overline{K}/K) \to GL(H^i(X_{\bar{K}}, \overline{\mathbb{Q}_\ell})).$$

We say that a representation which arises this way, or a subquotient of such a representation, *arises from geometry.* A more approachable version of Question 2.2.1 is:

Question 2.2.2. Which representations $\mathrm{Gal}(\overline{K}/K) \to GL_n(\overline{\mathbb{Q}_\ell})$ come from geometry?

If X is a complex variety, $x_0 \in X$ is a point, and $f : Y \to X$ is a smooth proper map, we obtain a monodromy representation

$$\pi_1(X^{\mathrm{an}}, x_0) \to GL(H^i(Y_{x_0}, L))$$

(for any coefficient field L). Again, we say that (subquotients of) representations of $\pi_1(X^{\mathrm{an}}, x_0)$ which arise this way *come from geometry.* The geometric analogue of Question 2.2.2 is:

Question 2.2.3. Which representations

$$\pi_1(X^{\mathrm{an}}, x_0) \to GL_n(L)$$

come from geometry? Analogously, for any field k of characteristic different from ℓ, and X a k-variety, which representations $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x_0}) \to GL_n(\overline{\mathbb{Q}_\ell})$ come from geometry?

Deligne has shown, via transcendental methods, that very few representations come from geometry.

Theorem 2.2.4 (Deligne, [Del87])**.** *Let X be a smooth Riemann surface and n a positive integer. Then the set of isomorphism classes of representations*

$$\pi_1(X^{an}, x_0) \to GL_n(\mathbb{Q})$$

which come from geometry is finite.

Note that this theorem is false if one replaces $\mathbb{Q}$ with $\overline{\mathbb{Q}}$; this is because subquotients of a representation defined over $\mathbb{Q}$ need not be defined over $\mathbb{Q}$. For example, taking $X = \mathbb{G}_m$, any finite order character of $\pi_1(X) = \mathbb{Z}$ comes from geometry — such characters are in bijection with roots of unity, and hence there are infinitely many of them.

I have proven a strengthening of Deligne's Theorem 2.2.4, which passes through number theory, rather than complex geometry:

Theorem 2.2.5 (L–, [Lit18a])**.** *Let k be an algebraically closed field of characteristic different from ℓ, X a smooth curve over k, $\bar{x}$ a geometric point of X, and L be a finite extension of $\mathbb{Q}_\ell$. Let n be a positive integer. Then the set of isomorphism classes of tame, semisimple representations*

$$\pi_1^{ét}(X, \bar{x}) \to GL_n(L),$$

which come from geometry, is finite.

This implies Theorem 2.2.4 by taking $k = \mathbb{C}$; even in this case, it is stronger than Theorem 2.2.4 because ℓ-adic fields contain algebraic extensions of $\mathbb{Q}$ of arbitrary degree.

The proof passes through anabelian geometry — the study of arithmetic fundamental groups of algebraic varieties. We make the following definition to explain the main result which implies Theorem 2.2.5.

Definition 2.2.6. Let k be a finitely generated field, and X a normal k-variety; let $\bar{x}$ be a geometric point of X. A semisimple, continuous representation

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

is said to be *arithmetic* if there exists a finite extension k' of k and a representation

$$\tilde{\rho} : \pi_1^{\text{ét}}(X_{k'}, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

with $\rho \simeq \tilde{\rho}|_{\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})}$.

The above definition makes sense because the natural map $X_{\bar{k}} \to X_{k'}$ induces by functoriality an inclusion of fundamental groups

$$\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \hookrightarrow \pi_1^{\text{ét}}(X_{k'}, \bar{x}).$$

Note that semisimple representations which come from geometry are in fact arithmetic.

Theorem 2.2.5 actually follows from the following stronger result:

Theorem 2.2.7 (L–, [Lit18a])**.** *Let X be a smooth curve over a finitely-generated field of characteristic different from ℓ; let $\bar{x}$ be a geometric point of X. Then the set of semisimple arithmetic representations*

$$\pi_1^{ét}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Q}_\ell})$$

is discrete.

That is, if $\{\rho_i\}_{i\in\mathbb{Z}_{>0}}$ is a sequence of semisimple arithmetic representations of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ with $\mathrm{Tr}(\rho_i)$ converging uniformly to $\mathrm{Tr}(\rho)$ in the ℓ-adic topology, then the sequence ρ_i is eventually constant. Note that Theorem 2.2.7 is a purely group-theoretic statement about the structure of arithmetic fundamental groups. But as a corollary, one may deduce the same discreteness statement for representations coming from geometry.

A compactness argument now gives:

Corollary 2.2.8 (L–, [Lit18a])**.** *Let k be an algebraically closed field of characteristic different from ℓ, X a smooth curve over k, $\bar{x}$ a geometric point of X, and L be a finite extension of $\mathbb{Q}_\ell$. Let n be a positive integer. Then the set of tame, semisimple arithmetic representations*

$$\pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to GL_n(L)$$

is finite.

Whence Theorem 2.2.5 follows, as representations which come from geometry are arithmetic. We view these results as analogues of the Shafarevich [Fal83] and Fontaine-Mazur finiteness conjectures [FM95] for function fields over algebraically closed fields.

Theorem 2.2.7 also implies a weak variant of the Frey-Mazur conjecture for function fields. We briefly recall the statement of the conjecture:

Conjecture 2.2.9 (Geometric Frey-Mazur conjecture [BT16])**.** *Let k be an algebraically closed field. There exists an integer $N = N(g, d)$ such that if C is a smooth curve of genus g and A_1, A_2 are geometrically traceless d-dimensional Abelian varieties over $k(C)$ with*

$$A_1[\ell] \simeq A_2[\ell]$$

as Galois representations, for some prime $\ell > N'$, then A_1 is $k(C)$-isogenous to A_2.

Our variant of the geometric Frey-Mazur conjecture is: if

$$\rho : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to GL_n(\overline{\mathbb{Z}_\ell})$$

is arithmetic and residually irreducible, there exists $N = N(\rho) \in \mathbb{Q}_{>0}$ such that any semisimple arithmetic representation $\tilde{\rho}$ with $\rho \simeq \tilde{\rho} \bmod \ell^N$ satisfies

$$\rho = \tilde{\rho}.$$

The Frey-Mazur conjecture suggests that one can compute N in terms of geometric invariants of X. Theorem 1.1.10 of [Lit18a] (building on work in [Lit18b]) gives an estimate for N; while the statement is complicated, it implies the following simple result:

Theorem 2.2.10 (L–, [Lit18a, Theorem 1.1.12])**.** *Let X be a normal complex variety. There exists a prime $\ell_0 = \ell_0(X)$ such that for any $\ell > \ell_0$, any representation*

$$\rho : \pi_1(X) \to GL_n(\overline{\mathbb{Z}})$$

which satisfies:

(1) $\rho \otimes \overline{\mathbb{Q}}$ is semisimple and comes from geometry, and
(2) is trivial mod ℓ

is trivial.

Again, this result follows from a similar result for semisimple arithmetic representations. Nadel and Hwang-To have proven related analytic results [Nad89, HT06].

Cadoret and Moonen [CM18] have recently begun building on this work — in particular, they proved a variant of Theorem 2.2.10.

2.2.2. *Lefschetz Theorems.* Given a projective variety X and an ample divisor $D \subset X$, a Lefschetz hyperplane theorem indicates when the map

$$F(X) \to F(D),$$

(or $F(D) \to F(X)$, if F is covariant) is an isomorphism (or injective/surjective) where F is some contravariant functor. For example, the classical Lefschetz hyperplane theorem considers the case where $F = H^i(-, \mathbb{Z})$ or $\pi_i(-)$. In SGA2 [Gro05], Grothendieck gives general techniques for answering these sorts of questions. The goal of [Lit18c], which grew out of my PhD thesis, was to extend the methods of SGA2 [Gro05] to study the case where F is a *representable* functor. One application of the main result of that paper [Lit18c, Theorem 1.10] is:

Theorem 2.2.11 (Non-abelian Lefschetz hyperplane theorem [Lit18c, Theorem 1.8])**.** *Let k be a field of characteristic zero, and X a smooth projective variety over k. Let $D \subset X$ be an ample Cartier divisor, and let Y be a smooth Deligne-Mumford stack over k with Ω^1_Y nef. Then the restriction functor*

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(D, Y)$$

is

- *fully faithful if* $\dim(Y) = \dim(D)$*, and* $\dim(X) \geq 2$
- *an equivalence if* $\dim(Y) < \dim(D)$ *and* $\dim(X) \geq 3$*.*

(Here $Hom(X, Y), Hom(D, Y)$ are groupoids, as Y is a stack.)

Letting $Y = BG$, where G runs over all finite groups, one recovers the Lefschetz hyperplane theorem for the étale fundamental group; but one may also take $Y = \mathscr{M}_g$ (the moduli space of smooth genus g curves), for example, to obtain a statement about extending families of curves off of ample divisors, among many other applications. The proof of Theorem 2.2.11 uses positive-characteristic vanishing techniques; indeed, it is a special case of a more general result [Lit18c, Theorem 1.10] which works in arbitrary characteristic, and with weaker hypotheses (i.e. Ω^1_Y need not be nef).

The other main contribution of [Lit18c] is the development of a "Lefschetz package" — a collection of deformation-theoretic and algebraization results designed to aid in the proof of Lefschetz theorems not covered by Theorem 2.2.11 above.

For example, Sommese gave a conjectural classification of all smooth projective varieties X containing a $\mathbb{P}^d$-bundle as an ample divisor [BS95, Conjecture 5.5.1]. This conjecture has been the subject of much interest among those working on the fine classification of algebraic varieties; see e.g. [Som76, Fuj80, B$\breve{8}$2a, B$\breve{8}$1, B$\breve{8}$2b, BFS05, FS88, FSS87, SS86, SS90, Sat88], and see [BI09] for a survey. In [Lit17] I applied the "Lefschetz package" of [Lit18c] to prove many new cases of Sommese's conjecture and to reduce it in general to a conjectural characterization of projective space. This characterization was almost immediately proved by Jie Liu [Liu16], thus resolving Sommese's conjecture:

Theorem 2.2.12 (Theorem 1.3 of [Liu16])**.** *Let X be a smooth projective variety and $Y \subset X$ a smooth ample divisor. Suppose that $p : Y \to Z$ is a morphism exhibiting Y as a $\mathbb{P}^d$-bundle over a b-dimensional manifold Z. Then one of the following holds:*

(1) $X \simeq \mathbb{P}^3$, $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ *is a smooth quadric, and p is one of the projections to $\mathbb{P}^1$.*
(2) $X \simeq Q^3 \subset \mathbb{P}^4$ *is a smooth quadric threefold,* $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ *is a hyperplane section, and p is a projection to one of the factors.*
(3) $Y \simeq \mathbb{P}^1 \times \mathbb{P}^b$, $Z \simeq \mathbb{P}^b$, $p : Y \to Z$ *is the projection to the second factor, and X is the projectivization of an ample vector bundle $\mathscr{E}$ on $\mathbb{P}^1$.*
(4) $X \simeq \mathbb{P}(\mathscr{E})$ *for an ample vector bundle $\mathscr{E}$ on Z, and* $\mathscr{O}_X(Y) \simeq \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ *(i.e. Y is a fiberwise hyperplane).*

2.2.3. *Dynamics.* In [LLar], my collaborator John Lesieutre and I used p-adic methods to analyze automorphism groups of algebraic varieties. The main goal of this paper was to understand how the automorphism group of a variety X changes under birational modifications of X. Our main result was:

Theorem 2.2.13 (Theorem 1.5 of [LLar])**.** *Let X be a smooth projective variety over $\mathbb{C}$ of dimension n, and let $Y = \mathrm{Bl}_Z(X)$, with exceptional divisor E, where Z is a smooth subvariety of X of dimension r. If either (1) $2r+3 \leq n$, or (2) $r+3 \leq n$ and $\mathrm{Nef}(E)$ is a polyhedral cone, then there exists an integer $N \geq 1$ such that for any automorphism ϕ of Y, one has that ϕ^N descends to X.*

Case (1) of this theorem was proven in [BC13] in the case that the Picard rank of X equals 1. This result is an extension of some of the results of [Les15] to varieties of arbitrary dimension.

2.2.4. *Mapping class group actions on character varieties.* Let $\Sigma_{g,n}$ be an orientable topological surface of genus g with n punctures, and let $MCG(g,n)$ be its mapping class group – that is, the fundamental group of the moduli space of genus g curves with n marked points. $MCG(g,n)$ has an outer action on $\pi_1(\Sigma_{g,n})$, and hence acts on the set of isomorphism classes of representations of $\pi_1(\Sigma_{g,n})$. A well-known question of Kisin, motivated by the p-curvature conjecture, asked:

Question 2.2.14 (Kisin)*.* Let
$$\rho : \pi_1(\Sigma_{g,n}) \to GL_m(\mathbb{C})$$
be a representation, whose isomorphism class has finite orbit under the natural action of $MCG(g,n)$ on isomorphism classes of representations described above. Is the image of ρ necessarily finite?

Almost immediately, it was observed by [BKMS18] that the answer to this question was negative. Nonetheless, Brian Lawrence and I prove [LL19]:

Theorem 2.2.15. *Suppose $2-2g-n<0$, and that for ever finite covering map $f : \Sigma_{g',n'} \to \Sigma_{g,n}$, the orbit of $f^*\rho$ under $MCG(g',n')$ is finite. Then ρ has finite image.*

This result suffices to give a reformulation of the p-curvature conjecture in terms of the so-called "non-Abelian Gauss-Manin connection" — as the p-curvature conjecture is already known for the usual Gauss-Manin connection [Kat72], one might hope to approach the p-curvature conjecture by imitating [Kat72] in the non-Abelian setting.

2.2.5. *Arithmetic of period domains.* Recent work of Lawrence-Venktesh [LV18] has given a new proof of the Mordell Conjecture — their techniques also give new results on non-density of integral points on varieties admitting a quasi-finite period map (i.e. a horizontal, locally liftable map to a period domain, with finite fibers). Motivated by their work, Ariyan Javanpeykar and I have proven the following finiteness and non-density result for points valued in finitely-generated $\mathbb{Z}$-algebras [JL19]:

Theorem 2.2.16. *Let $K \subset \mathbb{C}$ be a number field, S a set of places of K, and X a quasi-projective $\mathscr{O}_{K,S}$-scheme such that $X_{\mathbb{C}}$ admits a quasi-finite period map. Then the following are equivalent:*

1. *For all finite extensions K'/K, and all sets of places S' of k' containing all the places over S, the set of $\mathscr{O}_{K',S'}$ points of X is finite (resp. not Zariski dense in $X(\overline{\mathbb{Q}})$).*
2. *For all finitely-generated integral $\mathscr{O}_{K,S}$-algebras A, the set of A-points of X is finite (resp. not Zariski-dense in $X(\overline{\mathit{Frac}(A)})$).*

In other words, for varieties with quasi-finite period map, finiteness (resp. non-Zariski-density) of integral points persists over finitely generated extensions. The proof of the finiteness statement relies on Deligne's finiteness theorem (Theorem 2.2.4); the non-Zariski-density statement requires somewhat deeper Hodge theory, in particular, recent results of Bakker-Brunebarbe-Tsimerman [BBT18].

3. Proposed Research

The rest of this proposal will outline the several lines of research emanating from the work described above. The main thrust of this proposal is aimed at bringing aspects of the theory of arithmetic structures on fundamental groups, begun in [Lit16, Lit18b, Lit18a], to maturity, and pursuing many applications of that theory. Several of the results below are labeled "pre-theorems," meaning I have a detailed outline of a proof; "conjectures" are expected to be true, and in many cases I have a broad outline of a proof; and "questions" require more investigation before a precise mathematical statement can be made.

3.1. Anabelian geometry: applications to monodromy.

My research program at the moment has grown out of the work described in Section 2.2.1, and this proposal envisions applications far beyond those described there.

3.1.1. *ℓ-adic methods: around the geometric torsion conjecture and geometric Frey-Mazur conjecture.* One of the goals of the anabelian program sketched in this proposal is an approach to the geometric torsion conjecture, and the geometric Frey-Mazur conjecture, via ℓ-adic methods. We briefly recall these two conjectures here.

Conjecture 3.1.1 (Geometric torsion conjecture, [CT11, BT18])**.** *Let k be an algebraically closed field. There exists an integer $N = N(g, d)$ such that if C is a smooth curve of genus g over k and A is a traceless d-dimensional Abelian variety over $k(C)$, then*

$$\#|A(k(C))|_{\text{tors}} < N.$$

Conjecture 3.1.2 (Geometric Frey-Mazur conjecture [BT16])**.** *Let k be an algebraically closed field. There exists an integer $N' = N'(g, d)$ such that if C is a smooth curve of genus g and A_1, A_2 are traceless d-dimensional Abelian varieties over $k(C)$ with*

$$A_1[\ell] \simeq A_2[\ell]$$

as Galois representations, for some $\ell > N'$, then A_1 is isogenous to A_2.

The results of section 2.2.1 are in part motivated by these conjectures. I hope to prove both conjectures via variants of the techniques used to prove the results in section 2.2.1.

Goal 3.1.3. *Prove Conjectures 3.1.1 and 3.1.2.*

3.1.2. *An intermediate case.* I will first describe a strengthening of Theorem 2.2.10, which will serve as a template for a proposed proof of Conjecture 3.1.2.

Pre-Theorem 3.1.4. *There exists $N = N(q, g, \ell)$ such that if C is a smooth, geometrically connected curve (not necessarily projective) of genus g over $\mathbb{F}_q$, and*

$$\rho : \pi_1^{\acute{e}t}(C_{\overline{\mathbb{F}_q}}) \to GL_n(\overline{\mathbb{Z}_\ell})$$

is a representation such that:

1. *$\rho \otimes \overline{\mathbb{Q}_\ell}$ is semisimple arithmetic, and*
2. *$\rho \bmod \ell^N$ is trivial,*

then ρ is trivial.

This result is stronger than Theorem 2.2.10 in several ways, most notably that the uniformity is in the genus of C, and does not depend on C itself; in particular, one may take the same N for any open subset of C. As a corollary, we have

Corollary 3.1.5. *There exists $N = N(q, g, \ell)$ such that if C is a smooth, geometrically connected curve (not necessarily projective) of genus g over $\mathbb{F}_q$, and $A/\overline{\mathbb{F}_q}(C)$ is an Abelian variety with full level ℓ^N structure, then A is inseparably isogenous to an isotrivial Abelian variety.*

This is an analogue of the geometric torsion conjecture for Abelian schemes with full level structure over curves over finite fields; it is a positive-characteristic version of Nadel's theorem [Nad89]. It would be the first result of this type in positive characteristic. I will briefly sketch the proof of Pre-Theorem 3.1.4, indicate the obstructions to generalizing it to a proof of Conjectures 3.1.1 and 3.1.2, and then discuss how I expect to overcome those obstructions.

The key technical ingredient in the proof of Pre-Theorem 3.1.4 is the following result in arithmetic dynamics:

Pre-Theorem 3.1.6. *Let Γ be a topologically finitely-generated free pro-ℓ group, and $\varphi : \Gamma \xrightarrow{\sim} \Gamma$ a continuous automorphism such that:*

1. *(semisimplicity) φ acts semi-simply on the Lie algebra of the $\mathbb{Q}_\ell$-pro-unipotent completion of Γ, and*
2. *(the Siegel condition) if $\alpha_1, \cdots, \alpha_n$ are the eigenvalues of φ acting on $\Gamma^{ab} \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$, then (i) no nonempty monomial in the α_i equals 1, and (ii) there exists $\beta \geq 0, C > 0$ such that either*
$$\alpha_1^{i_1} \cdots \alpha_n^{i_n} - \alpha_j = 0$$
or
$$|\alpha_1^{i_1} \cdots \alpha_n^{i_n} - \alpha_j|_\ell \geq \frac{C}{|\sum_s i_s|^\beta}$$
for all j, and all $(i_1, \cdots, i_n) \in \mathbb{Z}^n_{\geq 0}$ with $\sum_s i_s \geq 2$.

Then there exists $N > 0$ such that if
$$\rho : \Gamma \to GL_m(\overline{\mathbb{Z}_\ell})$$
is a representation such that

1. *$\rho \otimes \overline{\mathbb{Q}_\ell}$ is semisimple,*
2. *$(\varphi^M)^*(\rho \otimes \overline{\mathbb{Q}_\ell}) \simeq \rho \otimes \overline{\mathbb{Q}_\ell}$ for some $M > 0$, and*
3. *ρ is trivial mod ℓ^N,*

then ρ is trivial. Here N depends only on C and β.

This result is a non-commutative ℓ-adic generalization of Siegel's linearization theorem (the commutative ℓ-adic variant of Siegel's linearization theorem is due to Herman and Yoccoz [HY83, §4]); its proof requires some fairly intricate ℓ-adic analysis.

Now let X be an smooth, affine, geometrically connected curve over $\overline{\mathbb{F}}_q$. To prove Pre-Theorem 3.1.4, we apply Pre-Theorem 3.1.6 by taking Γ to be the pro-ℓ geometric fundamental group of X, and φ a Frobenius automorphism of this group. Hypothesis (1) of Pre-Theorem 3.1.6 is satisfied by one of the main results of [Lit18b]; hypothesis (2) may be verified by combining the Weil conjectures for curves with the ℓ-adic form of Baker's theorem on linear forms in logarithms, due to Kunrui Yu [Yu07]. A fine analysis of the bounds coming from Yu's result show that one may take C, β to depend only on q, ℓ, and the genus of X.

3.1.3. *The general case.* While I think Pre-Theorem 3.1.4 is quite interesting, it does not suffice to prove Conjectures 3.1.1 and 3.1.2. There are two major obstructions to doing so via a generalization of the method proposed above. One may formulate an abstract theorem along the lines of Pre-Theorem 3.1.6 (replacing the pro-unipotent completion in the statement of the theorem with a slightly more complicated gadget, the relative pro-unipotent completion). However, the analogues of hypothesis (1) (semisimplicity) and hypothesis (2) (the Siegel condition) are more difficult to verify in this case. In [Lit18a] I explain how to verify the analogue of condition (1), conditional on the Tate conjecture (using the Langlands program for function fields as input); but condition (2) — as well as the Tate conjecture — seems to be out of reach at the moment. So it seems that a proof of the desired result over finite fields is hopeless by these methods.

That said, there does seem to be hope in characteristic zero. A good start would be a proof of the following conjecture, which at first blush may seem implausible:

Conjecture 3.1.7. *Fix a prime ℓ. Then there exists $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ such that for every Abelian variety $A/\mathbb{Q}$, there exists $N_A > 0$ such that σ^{N_A} acts on $T_\ell(A)$ via a homothety of infinite order.*

or the even more difficult:

Conjecture 3.1.8. *Fix a prime ℓ. Then there exists $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ such that for every representation*

$$\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_m(\overline{\mathbb{Q}_\ell})$$

which arises from geometry, there exists $N_\rho > 0$ such that $\rho(\sigma)^{N_\rho}$ is a homothety of infinite order.

That said, one may prove partial results towards these conjectures which are in my view encouraging. If one fixes ρ in Conjecture 3.1.8, the result is due to Bogomolov [Bog80]. Conjecture 3.1.7 in the case of elliptic curves follows from known uniform versions of Serre's open image theorem. In general, if one only allows A to vary in a family parametrized by a one-dimensional base, one may deduce the result from the main result of [CT13], as well as an analogous result towards Conjecture 3.1.8. I expect that the methods of [CT13] will give strong enough partial results for interesting applications. These results follow from an analogue of Pre-Theorem 3.1.6, taking Γ to be the geometric étale fundamental group of a curve and φ to be the Galois element σ provided by Conjecture 3.1.7 or 3.1.8.

For example, one can prove (using the results described in the previous paragraph) the following weak form of the geometric Frey-Mazur conjecture:

Pre-Theorem 3.1.9. *Let C be a smooth, geometrically connected curve over a finitely-generated field k of characteristic zero, $\bar{x}$ a geometric point of C, ℓ a prime, and $A \to C$ an Abelian scheme of relative dimension g. Let*

$$\rho : \pi_1^{\acute{e}t}(C_{\bar{k}}, \bar{x}) \to GL(T_\ell(A_{\bar{x}}))$$

be the associated monodromy representation. Then there exists $N(d)$ such that: if D is a set of points of C which each have degree at most d over k, and

$$\rho' : \pi_1^{\acute{e}t}(C_{\bar{k}} \setminus D, \bar{x}) \to GL_{2g}(\mathbb{Z}_\ell)$$

is semisimple arithmetic with $\rho|_{\pi_1^{\acute{e}t}(C_{\bar{k}} \setminus D, \bar{x})} \simeq \rho' \bmod \ell^{N(d)}$, then $\rho|_{\pi_1^{\acute{e}t}(C_{\bar{k}} \setminus D, \bar{x})}$ is isomorphic to ρ'.

It is not clear to me yet how accessible Conjecture 3.1.7 and 3.1.8 are in their full generality — that said, they do hold in general assuming strong forms of standard conjectures on uniform boundedness over *number fields.* So one reasonable outcome of this line of reasoning would be to prove the geometric torsion conjecture and geometric Frey-Mazur conjecture conditional on strong *arithmetic* uniform boundedness results.

3.1.4. *p-adic methods.* I have a different plausible program for a proof of Conjectures 3.1.1 and 3.1.2, building on the techniques of Section 2.2.1 — essentially, to augment the computations of the proof of Theorem 2.2.10 with input from integral p-adic Hodge theory.

As a toy example, I have, in unpublished work, proven:

Theorem 3.1.10 (Litt, unpublished). *Let K be a p-adic field and $X = \mathbb{P}^1_K \setminus \{x_1, \cdots, x_m\}$, for $x_1, \cdots, x_m \in \mathbb{P}^1_K(K)$. Then there exists $N = N(K)$ such that any non-trivial semisimple arithmetic representation*

$$\rho : \pi_1^{\acute{e}t}(X_{\bar{K}}) \to GL_n(\overline{\mathbb{Z}_p})$$

is non-trivial mod p^N.

Here the notion of *arithmeticity* of a representation is defined analogously to Definition 2.2.6. The proof uses some mild integral p-adic Hodge theory.

The main benefit of the proof is the following observation: if X has good reduction, the constant N in Theorem 3.1.10 may be bounded from above in terms of certain invariants of X — so-called p-adic iterated integrals, originally constructed by Coleman [Col82]. These are the p-adic analogue of complex (iterated) line integrals [Che71]. The key step in proving such a bound (which is not required for the proof of Theorem 3.1.10 itself) is an integral comparison theorem for the etale and log-crystalline fundamental groups for curves of good reduction.

Unfortunately there is not yet a sufficient theory of p-adic iterated integrals in the case of bad reduction (though Berkovich [Ber07] and Vologodsky [Vol03] have given inequivalent definitions) — in particular, for the applications I have in mind, one needs to go beyond the pro-unipotent setting. In joint work with Eric Katz, I will finish developing such a theory. An explicit consequence is a combinatorial description of Vologodsky integrals in terms of Berkovich integrals.

Goal 3.1.11. *Construct a theory of p-adic iterated integrals on semistable curves sufficient to achieve Goal 3.1.3, and give an efficient algorithm to compute p-adic iterated integrals on curves of bad reduction in terms of p-adic iterated integrals on (different) curves of good reduction.*

Unfortunately the exact definition of the Vologodsky and Berkovich integrals are too lengthy to include in this proposal — for experts, however, I include the following Pre-Theorem.

Pre-Theorem 3.1.12 (joint with Eric Katz)**.** *Let X be a curve over a p-adic field K and $\mathscr{X}$ its Berkovich analytification. Let $a, b \in \mathscr{X}$ be two points, and let $\omega_1, \cdots, \omega_n$ be elements of $H^0(X, \Omega^1_X)$. Then if $\mathscr{P}$ is the space of homotopy classes of paths from a to b, there exists an explicit function $w : \mathscr{P} \to K[[l(p)]]$ so that*

$$\int_a^b \omega_1 \cdots \omega_n = \sum_{p \in \mathscr{P}} w(p) \int_p^{Berk} \omega_1 \cdots \omega_n.$$

Here $l(p)$ is a (formal) branch of the p-adic logarithm, the integral on the left is the Vologodsky iterated integral, and the integral on the right is the Berkovich iterated integral. The function w can be computed in terms of the combinatorics of the skeleton of $\mathscr{X}$.

Remark 3.1.13. I expect this foundational work on p-adic iterated integrals in bad reduction will have many applications aside from Goal 3.1.3 — in particular, it should be useful for applying the non-Abelian Chabauty method for finding rational points on curves in cases of bad reduction.

The first key step in proving of Conjectures 3.1.1 and 3.1.2 would be:

Goal 3.1.14. *Bound the p-adic valuation of p-adic iterated integrals on $X \setminus D$, where X is a smooth proper curve (not necessarily of good reduction), and $D \subset X$ is a divisor.*

I expect that this is doable.

The second key step is an integral comparison theorem for the etale and log-crystalline fundamental groups of curves with not-necessarily good reduction. I already know how to prove such a comparison theorem, but the method is somewhat artificial — essentially an integral version of [KH04, KH05]. New work of Bhatt-Scholze [BS19] on *prismatic cohomology* (a family of p-adic cohomology theories for algebraic varities in positive characteristic) suggests a more natural approach:

Goal 3.1.15. *Develop a theory of* prismatic fundamental groups *which specializes to the pro-unipotent de Rham, crystalline, and p-adic étale fundamental groups.*

Morrow and Tsuji have some ongoing related work (yet to appear in print).

3.2. **Anabelian methods: applications to rational points.**

3.2.1. *Around the section conjecture.* Perhaps the best-known conjecture in anabelian geometry is Grothendieck's section conjecture, which we now recall.

Conjecture 3.2.1 (Grothendieck's section conjecture)**.** *Let C be a smooth, proper, geometrically connected curve of genus $g \geq 2$ over a number field k, and let $\bar{x}$ be a geometric point of C. Then the set of rational points $C(k)$ is naturally in bijection with the set of conjugacy classes of splittings of the short exact sequence*

$$1 \to \pi_1^{\acute{e}t}(C_{\bar{k}}, \bar{x}) \to \pi_1^{\acute{e}t}(C, \bar{x}) \to Gal(\bar{k}/k) \to 1. \tag{3.2.2}$$

There are very few cases where this conjecture is known; to my knowledge, the only known examples have been constructed by Harari-Szamuely [HS09] and Stix [Sti13].

Goal 3.2.3. *Produce new examples of curves satisfying the conclusions of Grothendieck's section conjecture.*

The easiest way to produce such examples is to produce curves C/k so that sequence 3.2.2 does not admit *any* splittings (and hence such that $C(k)$ is empty). We now discuss techniques by which one may produce such examples.

3.2.2. *An intermediate case: abelian obstructions.* Pushing out sequence 3.2.2 along the quotient map $\pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x}) \to \pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x})^{\text{ab}}$, one obtains a squence

$$1 \to \pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x})^{\text{ab}} \to W \to \text{Gal}(\bar{k}/k) \to 1 \tag{3.2.4}$$

for some pro-finite group W. If sequence 3.2.4 does not split, then the same is true for sequence 3.2.2.

In joint work with Padmavathi Srinivasan, I will prove:

Pre-Theorem 3.2.5 (joint with Padmavathi Srinivasan)**.** *For each $g \geq 6$, there exist infinitely many pairs (C, k) where k is a number field and C is a smooth, projective, geometrically connected curve over k, such that sequence 3.2.4 does not split.*

In particular, these give examples where the section conjecture is true in every genus $g \geq 6$. In fact, I expect the stronger result to follow from our methods for $g = 6$:

Pre-Theorem 3.2.6 (joint with Padmavathi Srinivasan)**.** *For every number field k, there exist infinitely many smooth, projective, geometrically connected curves of genus 6 such that sequence 3.2.4 does not split.*

I'll briefly sketch the proof of Pre-Theorems 3.2.5 and 3.2.6, and then discuss how we plan to extend it in more sophisticated directions. The main idea has two steps: (1) a topological computation, and (2) a specialization argument.

The obstruction to splitting sequence 3.2.4 is a class in $H^2(k, \pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x})^{\text{ab}})$; letting $\pi : \mathscr{C}_g \to \mathscr{M}_g$ be the universal genus g curve over the moduli stack of genus g curves over $\mathbb{Q}$, we see that this class is pulled back from a class

$$o \in H^2(\mathscr{M}_g, (R^1\pi_*\widehat{\mathbb{Z}})^\vee).$$

A topological computation, essentially due to Morita [Mor86], shows that this class does not vanish (though to apply our method we must combine Morita's computation with related work of Hain [Hai95] and Ebert-Randal-Williams [ERW12]) if $g \geq 6$, though it is torsion. After some non-trivial analysis of how this class degenerates on the boundary of $\mathscr{M}_g$, one may show that there are infinitely many closed points x of $\mathscr{M}_g$ so that x^*o is non-zero, using methods analogous to the specialization results of Fein-Saltman-Schacher for Brauer classes [FSS92]. The stronger result in Pre-Theorem 3.2.6 follows via a variant specialization argument, using that $\mathscr{M}_6$ is rational.

3.2.3. *Non-abelian obstructions.* The results described in the preceding section are unsatisfying in the sense that the obstructions to splitting sequence 3.2.2 are *abelian*. In particular, not only do none of the curves produced by Pre-Theorems 3.2.5 and 3.2.6 have a rational point — none of them have a degree 1 zero-cycle.

Goal 3.2.7. *Produce infinitely many (C, K) where K is a number field, C is a smooth, geometrically connected curve over K, and such that sequence 3.2.4 splits but sequence 3.2.2 does not.*

In other words, we want to produce examples where the section conjecture is verified, but for non-abelian reasons.

I have a plausible program to produce such examples, following a more sophisticated version of the method described in the previous section. As before, let $\mathscr{M}_g$ be the moduli stack of curves of genus g, and $\pi : \mathscr{C}_g \to \mathscr{M}_g$ the universal curve. Let $\mathrm{Pic}^1_{\mathscr{C}_g/\mathscr{M}_g}$ be the (relative) moduli space of degree 1 line bundles on the universal curve. Then a geometric interpretation of the class o described above shows (almost tautologically) that

$$o|_{\mathrm{Pic}^1_{\mathscr{C}_g/\mathscr{M}_g}} = 0,$$

so for any k-point of $\mathrm{Pic}^1_{\mathscr{C}_g/\mathscr{M}_g}$, (corresponding to a curve C/k and a line bundle $\mathscr{L}$), the sequence 3.2.4 associated to C splits.

Now we consider the sequence

$$1 \to \pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x})/[[\pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x}), \pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x})], \pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x})] \to W' \to \mathrm{Gal}(\bar{k}/k) \to 1 \tag{3.2.8}$$

obtained by pushing out sequence 3.2.2 along the natural quotient map

$$\pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x}) \to \pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x})/[[\pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x}), \pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x})], \pi_1^{\text{ét}}(C_{\bar{k}}, \bar{x})].$$

In cases where sequence 3.2.4 splits, the splitting (or lack thereof) of sequence 3.2.8 is also controlled by a certain cohomological obstruction (namely, a variant of the Massey product).

Goal 3.2.9. *Show, via a topological computation, that the universal such obstruction (on $\mathrm{Pic}^1_{\mathscr{C}_g/\mathscr{M}_g}$) does not vanish.*

Remark 3.2.10. It is not hard to see that once again this universal obstruction is torsion.

I expect this goal to be doable but highly non-trivial — the analogous computations required for Pre-Theorems 3.2.5 and 3.2.6 are already quite involved. That said, I expect this computation will yield to techniques similar to those in [Mor86] or [ERW12].

Goal 3.2.11. *Show that there are infinitely many points of $\mathrm{Pic}^1_{\mathscr{C}_g/\mathscr{M}_g}$ at which the desired obstruction class specializes to something nonzero.*

I expect this to work essentially identically to the analogous specialization step in the proof of Pre-Theorems 3.2.5 and 3.2.6.

3.2.4. *p-adic methods: the non-Abelian Chabauty method.* One of the most important results in arithmetic geometry is Faltings's Theorem (originally the Mordell conjecture):

Theorem 3.2.12 ([Fal83])**.** *Let X be a curve of genus at least 2 over a number field k. Then $X(k)$ is finite.*

We currently have no way of computing the finite set $X(k)$ in general, or even of bounding its size. Conjecturally, $\#|X(k)|$ is bounded uniformly in terms of the genus of X and the degree of k:

Conjecture 3.2.13 (Uniform Mordell Conjecture)**.** *There exists $N = N(d, g)$ such that if k is a number field of degree d and X is a curve of genus g over k, then*

$$\#|X(k)| < N.$$

This conjecture is completely out of reach. The existing results in this direction (e.g. [KRZB16]) assume that the Jacobian of X has small rank and use the Chabauty method at a small prime p. To achieve uniform results, one must not assume X has good reduction at p. Therefore, I expect one output of Goal 3.1.11 will be applications to (variants of) the non-Abelian Chabauty-Kim method, executed at primes of bad reduction, which would conjecturally compute all rational points on X. Unfortunately, there are serious technical obstructions at the moment to running the non-Abelian Chabauty-Kim method at primes of bad reduction.

Let k_p be the completion of k at a p-adic place. Broadly speaking, the idea of the non-Abelian Chabauty-Kim method is to use certain p-adic analytic maps from X to certain so-called Selmer varieties:

$$\iota_n : X_{k_p}^{\mathrm{an}} \hookrightarrow \mathrm{Sel}_n.$$

These Selmer varieties are closely connected to the Galois action on the étale fundamental group of X, and to the crystalline fundamental group of the reduction of X at p. Kim's idea was to find certain p-adic analytic functions on Sel_n which vanish at k-rational points of X. One of the main obstructions to using the Chabauty-Kim method at places of bad reduction is that the structure of these Selmer varieties is poorly understood at bad primes.

Goal 3.2.14. *Use the non-Abelian Chabauty-Kim method* [Kim18] *at small primes to bound the number of rational points on some interesting families of curves.*

In joint work with Alexander Betts, I have already made some significant progress towards developing the Chabauty-Kim method for use at primes of bad reduction. The key technical input is the following semisimplicity result for Galois actions on fundamental groups of varieties with no good reduction assumptions, building on work in [Lit18b]:

Pre-Theorem 3.2.15 (joint with Alexander Betts)**.** *(1) Let X be a smooth variety over a p-adic field K, and let ℓ be a prime different from p. Let φ be a Frobenius element of $Gal(\overline{K}/K)$, $x \in X(K)$ a rational point, and $\pi_1^{\ell}(X_{\bar{K}}, \bar{x})^{unip}$ the $\mathbb{Q}_\ell$-pro-unipotent fundamental group of $X_{\bar{K}}$. Then φ acts semi-simply on Lie $\pi_1^{\ell}(X_{\bar{K}}, \bar{x})^{unip}$.*

(2) Let X be a proper semistable variety over the finite $\mathbb{F}_{p^k}$, and endow it with the canonical log structure X^{log} making it smooth over the log point. Fix a rational point of X and let $\pi_1^{crys,\ unip}$ be the associated pro-unipotent log-crystalline fundamental group. Then if φ is the crystalline Frobenius, φ^k acts semi-simply on Lie $\pi_1^{crys,\ unip}$.

The main input comes from (known cases of) the weight monodromy conjecture. One may (non-trivially) deduce the following consequence.

Corollary 3.2.16. *The Selmer varieties appearing in the Chabauty-Kim method are (non-canonically) isomorphic to affine spaces, with explicitly computable dimension.*

This is perhaps the zero-th step in running the Chabauty-Kim method at bad primes.

3.3. **Complements.** In this section I will describe two other projects which naturally flow from the work described above, but which are slightly less developed at present.

3.3.1. *Complex-analytic complements.* The work described in Section 3.1.4 has a natural complex-analytic analogue. Let X be a smooth manifold and $\gamma : [0,1] \to X$ a smooth map. Let

$$p_i : [0,1]^n \to [0,1], i = 1, \cdots n$$

be the projection maps. Then if $\omega_1, \cdots, \omega_n$ are 1-forms on X, the iterated integral $\int_\gamma \omega_1 \cdots \omega_n$ is defined to be the integral

$$\int_\gamma \omega_1 \cdots \omega_n := \int_{0 \le t_1 \le t_2 \cdots \le t_n \le 1} (\gamma \circ p_1)^* \omega_1 \wedge \cdots \wedge (\gamma \circ p_n)^* \omega_n.$$

This is a fancy way of discussing integrals of the form

$$\int_0^1 g(t_2) \int_0^{t_2} f(t_1) dt_1 dt_2,$$

i.e. multiple integrals where a variable being integrated over appears in the limits of integration. In contrast to the p-adic situation, where iterated integrals only depend on the endpoints, this form of iterated integral depends crucially on the path γ.

I recently noticed that if X is a smooth complex variety, there is a variant of iterated integration which is path independent, and which arises from the mixed Hodge structure on the fundamental group(oid) of X. Let x be a point of X, and let $\mathbb{C}[[\pi_1(X, x)]]$ be the completion of the group ring of the (topological) fundamental group of X at its augmentation ideal. If y is another point of X, let $\mathbb{C}[[\pi_1(X, x, y)]]$ be the free left $\mathbb{C}[[\pi_1(X, x)]]$-module generated by a (choice of) path from x to y. There is a natural mixed Hodge structure on $\mathbb{C}[[\pi_1(X, x, y)]]$, due to Chen [Che71], Morgan [Mor78], Hain [Hai87], and others.

Pre-Theorem 3.3.1 (L–, unpublished)**.** *Let X be a smooth, connected complex algebraic variety, and let $x, y \in X$. Then there exists a unique element $p(x, y) \in \mathbb{C}[[\pi_1(X; x, y)]]$, such that:*

(1) $\epsilon(p(x, y)) = 1$,

(2) $p(x, y) \in F^0\mathbb{C}[[\pi_1(X; x, y)]]$, *and*

(3)

$$\overline{p(x, y)} \in \bigcap_{i>0} (W^{-i-1} + F^{-i+1}).$$

Her $F^\bullet$ is the Hodge filtration, $W^\bullet$ is the weight filtration, $\epsilon : \mathbb{C}[[\pi_1(X, x, y)]] \to \mathbb{C}$ is the augmentation map, and $\overline{p(x, y)}$ is the complex-conjugate of $p(x, y)$.

Corollary 3.3.2. *Any unipotent flat vector bundle on a smooth, pointed complex variety X has a canonical (functorial) real-analytic trivialization.*

One may make sense of iterated integrals along the paths $\overline{p(x, y)}$, giving *single-valued* iterated integrals on complex varieties. For example, if $X = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$, and $\omega = \frac{dz}{z}$, we have

$$\int_a^b \omega = 2 \log |b/a|.$$

More generally, these single-valued iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ recover well-known special functions, including the single-valued Bloch-Wigner-Ramakrishnan polylogarithms (discussed in e.g. [Zag90]), and make the functional equations (and other properties) of these functions transparent. In my one main selling point of Pre-Theorem 3.3.1 is to give these special functions a geometric meaning.

Goal 3.3.3. *Develop the complex-analytic theory of single-valued iterated integrals.*

3.3.2. *Around the p-curvature conjecture and the mapping class group.* I would like to continue the work begin in [LL19] around the p-curvature conjecture, described in Section 2.2.4. As in Section 3.2, let $\mathscr{M}_g$ be the moduli stack of genus g curves, and $\mathscr{C}_g \to \mathscr{M}_g$ the universal curve of genus g. Lawrence and I give, in [LL19], a reformulation of the p-curvature conjecture in terms of the geometry of a certain foliation on the moduli space of flat vector bundles on the universal curve; this foliation is the non-Abelian analogue of the Gauss-Manin connection.

One reason to think that this is progress, rather than empty formalism, is that the p-curvature conjecture was proven by Katz in the case of the usual Gauss-Manin connection [Kat72]. While my belief is that the p-curvature conjecture is still well out of reach, it is natural to investigate parts of Katz's proof that may be extended to the non-Abelian setting. Less ambitiously and more practically, variants of my work with Lawrence suggest that one may be able to prove cases of the p-curvature conjecture for the *generic curve*, as in work of Patel-Shankar-Whang [PSW18] for rank 2 vector bundles. In joint work with Brian Lawrence and Peter Whang, I hope to prove:

Goal 3.3.4. *Fix an integer n and let $g \gg n$. Then the p-curvature conjecture is true for rank n vector bundles on the generic curve of genus g.*

This result is likely accessible by a variant of the methods applied in [LL19], combined with the ideas of [PSW18].

4. Broader impacts

I am committed to the broad dissemination of mathematical knowledge, both within the professional mathematical community and to the public at large. Many of the questions here have aspects suitable for undergraduate research or beginning graduate research — as I have in past years, I plan to continue supervising undergraduate research projects. For example, variants of the work described in Section 3.3.1 are accessible to any undergraduate with a background in multi-variable calculus.

Of course I will continue to disseminate my own research both domestically and abroad, and to collaborate with mathematicians around the world, and to organize conferences related to the work described in this proposal. I plan to organize a conference on anabelian techniques in the coming years. I plan to write educational materials about these ongoing projects — my mathematical blog is already active and well-trafficked, and I hope to grow it to an important source of mathematical interaction and collaboration. I also plan to continue more public science communication efforts online, as in [num19]. Finally, I plan to continue giving expository talks to students at all levels (graduate, undergraduate, high school, and below) about mathematics and the joys of pursuing it.

5. Intellectual Merit

The obvious sense in which this proposal will advance knowledge within algebraic geometry and number theory is through progress towards a proof of the conjectures listed. In particular, I think proofs of (generalizations of) Conjecture 3.1.1 (the geometric torsion conjecture) and Conjecture 3.1.2 (the geometric Frey-Mazur conjecture) are within reach, given the techniques developed here. The constructions described in section 3.2 will provide important evidence for the section conjecture. And moreover the techniques described here themselves will have applications throughout algebraic geometry and number theory — in particular, I expect the work on integral p-adic Hodge theory of fundamental groups and on p-adic iterated integrals in bad reduction to have important applications to Diophantine questions (via variants of the Chabauty-Kim method).

More generally, this proposal includes several aspects which I believe have the potential to dramatically advance our knowledge. In particular, the anabelian nature of the proposed approaches to the geometric torsion conjecture and geometric Frey-Mazur conjecture means that the ideas here can in principle be imitated in attempting a proof of the torsion conjecture or Frey-Mazur conjecture over *number fields*, rather than function fields — this is an area where progress has essentially stalled.

I think the interdisciplinary nature of the proposed projects is also very exciting; for example, the proposed work on iterated integration promises to connect ideas in complex and p-adic geometry. Moreover, the proposed work on both the section conjecture and the p-curvature conjecture promises to open new and (in my view) extremely appealing interactions between geometric topology and arithmetic geometry.

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