ISOMONODROMY, STABILITY, AND HODGE THEORY

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ABSTRACT. We show that the minimum rank of a non-isotrivial geometric local system on a suitably general n-pointed curve of genus g, with unipotent monodromy at infinity, is at least $2\sqrt{g+1}$. The main input is an analysis of stability properties of flat vector vector bundles under isomonodromic deformation, which additionally answers a question of Biswas, Heu, and Hurtubise.

1. INTRODUCTION

1.1. **Overview.** We work over the complex numbers $\mathbb{C}$. The main result of this paper, Theorem 1.2.4, is that an analytically very general n-pointed curve of genus g does not carry any non-isotrivial polarizable $\mathbb{Z}$-variations of Hodge structure with unipotent monodromy around the marked points, of rank less than $2\sqrt{g+1}$. In particular, an analytically very general smooth proper curve of genus g carries no geometric local systems of rank less than $2\sqrt{g+1}$ with infinite monodromy, as we show in Corollary 1.2.6. This is a strong restriction on the topology of smooth proper maps to an analytically very general curve. (See Definition 1.2.2 for the definition of "analytically very general.")

These results rely on an analysis of stability properties of isomonodromic deformations of flat vector bundles with regular singularities, and require correcting a number of errors in the literature on this topic.

Let C_0 be the central fiber of a family of curves $\mathscr{C} \to \Delta$ with Δ a contractible domain, and let (E_0, ∇_0) be a flat vector bundle on C_0. Recall that, loosely speaking, the isomonodromic deformation of (E_0, ∇_0) is the deformation $(\mathscr{E}, \nabla)$ of (E_0, ∇_0) to $\mathscr{C}/\Delta$, such that the monodromy of the connection is constant.

In Corollary 1.3.3, we construct a flat vector bundle on a smooth proper curve over $\mathbb{C}$, whose isomonodromic deformations to a nearby curve are never semistable. (See Definition 2.2.4 for precise definitions.) The construction arises from the "Kodaira-Parshin trick," and contradicts earlier claimed theorems of Biswas, Heu, and Hurtubise ([BHH16, Theorem 1.3], [BHH21, Theorem 1.3], and [BHH20, Theorem 1.2]), which imply that such

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a construction is impossible. See Remark 4.1.9 for a discussion of the errors in those papers.

As a complement to this example, we show in Theorem 1.3.4 that any logarithmic flat vector bundle admits an isomonodromic deformation to a nearby curve which is *close* to semistable, in a suitable sense, and moreover is semistable if the rank is small compared to the genus of the curve. While our results contradict those of [BHH16, BHH21, BHH20], our methods owe those papers a substantial debt. Biswas, Heu, and Hurtubise pitch the question of isomonodromically deforming a vector bundle to a semistable vector bundle (see Question 1.3.1) as an analogue of Hilbert's 21st problem, also known as the Riemann-Hilbert problem.

This semistability property is also the main input to the above mentioned Hodge theoretic results. The applications to polarizable variations of Hodge structures come from the fact that flat vector bundles underlying polarizable variations are rarely semistable, due to well-known curvature properties of Hodge bundles.

1.2. **Main Hodge-theoretic results.** For convenience, throughout the paper, out main results will primarily be stated for hyperbolic curves.

Definition 1.2.1. Let C be a curve over $\mathbb{C}$ of genus g and $D \subset C$ a reduced effective divisor of degree n. Call (C, D) *hyperbolic* if C is a smooth proper connected curve and either $g \geq 2$, $g = 1$ and $n > 0$, or $g = 0$ and $n > 2$. We call an n-pointed curve $(C, x_1, \ldots, x_n)$ *hyperbolic* if $(C, x_1 + \cdots + x_n)$ is hyperbolic.

Equivalently, (C, D) is hyperbolic if it has no infinitesimal automorphisms, i.e., $H^0(C, T_C(-D)) = 0$.

We will also work with the following analytic notion of a (very) general general point.

Definition 1.2.2. A property holds for an *analytically general* point of a complex orbifold X, if there exists a nowhere dense closed analytic subset $S \subset X$ so that the property holds on $X - S$. We say that a property holds for an *analytically very general* point if, locally on X, there exists a countable collection of nowhere dense closed analytic subsets such that the property holds on the complement of their union. If $\mathscr{M}_{g,n}$ is the analytic moduli stack of n-pointed curves of genus g, we say that a property holds for an analytically (very) general n-pointed curve if it holds for an analytically (very) general point of $\mathscr{M}_{g,n}$.

Remark 1.2.3. From the definition, it may appear that "analytically very general" is a local notion, while "analytically general" is a global notion. However, being "analytically general" also has the following equivalent local definition, which is more similar to the definition of "analytically very general": locally on X, there exists a nowhere dense closed analytic subset such that the property holds on the complement of this subset.

The main geometric consequence of this work is the following constraint on the rank of non-isotrivial polarizable variations of Hodge structure (defined in §3) on an analytically very general curve:

Theorem 1.2.4. *Let K be a number field with ring of integers $\mathscr{O}_K$. Then for an analytically very general n-pointed hyperbolic curve $(C, x_1, \cdots, x_n)$ of genus g, if $\mathbb{V}$ is a $\mathscr{O}_K$-local system on $C \setminus \{x_1, \cdots, x_n\}$ with infinite monodromy, and unipotent monodromy about the x_i, such that for each embedding $\iota : \mathscr{O}_K \to \mathbb{C}$, $\mathbb{V} \otimes_{\mathscr{O}_K, \iota} \mathbb{C}$ underlies a polarizable complex variation of Hodge structure, we have*

$$\mathrm{rk}_{\mathscr{O}_K}(\mathbb{V}) \geq 2\sqrt{g+1}.$$

Remark 1.2.5. Note that a result analogous to Theorem 1.2.4 does not hold for variations without an underlying $\mathscr{O}_K$-structure. Indeed, every smooth proper curve of genus at least 2 admits a non-unitary complex polarizable variation of Hodge structure of rank 2, arising from uniformization (see e.g. [Sim88, bottom of p. 870]).

As local systems which arise from geometry satisfy the hypotheses of Theorem 1.2.4, we have:

Corollary 1.2.6. *Let $(C, x_1, \cdots, x_n)$ be an analytically very general hyperbolic n-pointed curve of genus g. If $f : X \to C \setminus \{x_1, \cdots, x_n\}$ is a smooth proper algebraic morphism, $i \geq 0$ is an integer, and $\mathbb{V} \subset R^i f_* \mathbb{C}$ is a sub-local system with infinite monodromy and unipotent monodromy about the x_i, then $\dim_{\mathbb{C}} \mathbb{V} \geq 2\sqrt{g+1}$.*

For example, we have the following concrete geometric corollary:

Corollary 1.2.7. *If $(C, x_1, \ldots, x_n)$ is an analytically very general hyperbolic n-pointed genus g curve, then any non-isotrivial abelian scheme over $C \setminus \{x_1, \cdots, x_n\}$ with semistable reduction at the x_i has relative dimension at least $\sqrt{g+1}$.*

Proof. This follows by applying Corollary 1.2.6 to the map f giving the relative abelian scheme, with $i = 1$. The non-isotriviality condition implies the monodromy is infinite and having semistable reduction implies the monodromy about the x_i is unipotent [Gro72, Exp IX, 3.5]. □

In what follows, we say a flat vector bundle has *unitary monodromy* if the associated monodromy representation $\rho : \pi_1(C) \to \mathrm{GL}_n(\mathbb{C})$ has image with compact closure. We will deduce the above results from Theorem 1.2.8 below, using that a discrete subset of the image of a unitary ρ is finite.

Theorem 1.2.8. *Let $(C, x_1, \cdots, x_n)$ be an n-pointed hyperbolic curve of genus g. Let (E, ∇) be a flat vector bundle on C with $\operatorname{rk} E < 2\sqrt{g+1}$ and with regular singularities and nilpotent residues at the x_i. If an isomonodromic deformation of E to an analytically general nearby n-pointed curve underlies a polarizable complex variation of Hodge structure, then (E, ∇) has unitary monodromy.*

1.3. **Main results on isomonodromic deformations.** As remarked in § 1.1, the Hodge-theoretic results of § 1.2 arise from an analysis of the Harder-Narasimhan filtrations of isomonodromic deformations of flat vector bundles on curves. Our first such result is a counterexample to [BHH16, Theorem 1.3], [BHH21, Theorem 1.3], and [BHH20, Theorem 1.2], which demonstrates that the situation is somewhat more complicated than was previously believed — there exist irreducible flat vector bundles whose isomonodromic deformations are never semistable.

Specifically, [BHH16] ask the following question.

Question 1.3.1 ([BHH16, p. 123])**.** Let X be a smooth proper curve, and $D \subset X$ a reduced effective divisor. Given a flat vector bundle (E, ∇) on X, with regular singularities along D, let (E', ∇') be the isomonodromic deformation of (E, ∇) to an analytically general nearby curve (X', D'). Is E' semistable?

The main claim of [BHH16] is that Question 1.3.1 has a positive answer if (E, ∇) has irreducible monodromy and the genus of X is at least 2. However, the following results answer Question 1.3.1 in the negative, even in this case. See Remark 4.1.9 for a discussion of the errors in previous claims that Question 1.3.1 had a positive answer.

We use $\mathscr{M}_{g,n}$ to denote the analytic moduli stack of smooth proper curves with geometrically connected fibers and n distinct marked points. Let $\mathscr{T}_{g,n}$ denote the universal cover of $\mathscr{M}_{g,n}$, and let $\tau : \mathscr{X} \to \mathscr{T}_{g,n}$ denote the universal curve.

Theorem 1.3.2. *Let $g \geq 2$ be an integer. There exists a vector bundle with flat connection $(\mathscr{F}, \nabla)$ on $\mathscr{M}_{g,1}$ such that for each fiber C of the forgetful morphism $\mathscr{M}_{g,1} \to \mathscr{M}_g$, the restriction of $(\mathscr{F}, \nabla)$ to C*

(1) has semisimple monodromy and
(2) is not semistable.

We also have the following variant, where the vector bundle has irreducible monodromy, instead of just semisimple monodromy.

Corollary 1.3.3. *Let C be a smooth projective curve of genus at least 2. There exists an irreducible flat vector bundle (E, ∇) on C, whose isomonodromic deformations to a nearby curve are never semistable.*

Proof. The restriction $(\mathscr{F}, \nabla)|_C$ from Theorem 1.3.2 provides a semisimple flat vector bundle; by Theorem 1.3.2(2), its isomonodromic deformation to a nearby curve is never semistable. Hence one of the irreducible summands of $(\mathscr{F}, \nabla)|_C$ satisfies the statement of the corollary. □

In a positive direction, we have have the following result, showing that the isomonodromic deformation of any semisimple flat vector bundle to an analytically general nearby curve is close to being semistable, and moreover it is semistable if the rank is small.

Theorem 1.3.4. *Let (C, D) be hyperbolic and let (E, ∇) be a flat vector bundle on C with regular singularities along D and irreducible monodromy. Suppose (E', ∇') is an isomonodromic deformation of (E, ∇) to a general nearby curve, with Harder-Narasimhan filtration $0 = F'^0 \subset F'^1 \subset \cdots \subset F'^n = E'$, for $1 \leq i \leq n$. Let μ_i denote the slope of $\mathrm{gr}^i_{HN} E' := F'^i / F'^{i-1}$. Then the following two properties hold.*

(1) If E' is not semistable, then for every $0 < i < n$, there exists $j < i < k$ with

$$\mathrm{rk}\,\mathrm{gr}^{j+1}_{HN} E' \cdot \mathrm{rk}\,\mathrm{gr}^{k}_{HN} E' \geq g + 1.$$

(2) We have $0 < \mu_i - \mu_{i+1} \leq 1$ for all $i < n$.

In other words, the consecutive associated graded pieces of the generic Harder-Narasimhan filtration have slope differing by at most one, and, if there are multiple pieces of the generic Harder-Narasimhan filtration, many of them must have large rank relative to g.

Remark 1.3.5. Theorem 1.3.4 also holds without the hyperbolicity assumption, as we will explain. Nevertheless, it is convenient to make the assumption so that curves have no infinitesimal automorphisms. In this case isomonodromic deformations are somewhat better behaved, see [Heu10, p. 518].

We now explain the proof of Theorem 1.3.4 in the case (C, D) is not hyperbolic. Suppose (C, D) is not hyperbolic, so either $g = 1, n = 0$ or $g = 0, n \leq 2$. In this case the fundamental group $\pi_1(C - \{x_1, \ldots, x_n\})$ is abelian. This implies any irreducible representation of $\pi_1(C - \{x_1, \ldots, x_n\})$ is 1-dimensional, so the corresponding flat vector bundle is a line bundle. In this case, E and E' are semistable, so Theorem 1.3.4 still holds.

As a corollary, we are able to salvage the main theorem of [BHH16] for flat vector bundles whose rank is small relative to g, using the AM-GM inequality.

Corollary 1.3.6. *Let (C, D) be a hyperbolic curve of genus g. Let (E, ∇) be a flat vector bundle with regular singularities along D, and suppose that* $\mathrm{rk}(E) <$

$2\sqrt{g+1}$. *Then an isomonodromic deformation of* (E, ∇) *to an analytically general nearby curve is semistable.*

Our methods are heavily inspired by those of [BHH16], but our technique requires some new input from Clifford theory for vector bundles.

1.4. **Motivation.** Our main motivation comes from the following question. Let $f : X \to Y$ be a map of algebraic varieties. What are the restrictions on the topology of f? Our Corollary 1.2.6 places a very strong restriction on the topology of morphisms to an analytically very general curve C of genus g. For example, it implies that if $f : X \to C$ is a strictly semistable family with smooth generic fiber and bad reduction at n analytically very general points of C, then any non-isotrivial monodromy representation occurring in the cohomology of X/C has dimension at least $2\sqrt{g+1}$.

We became interested in this question and its connection to isomonodromy while trying to understand [BHH16]. In that paper Biswas, Heu, and Hurtubise raise Question 1.3.1, asking whether it is possible to isomonodromically deform irreducible flat vector bundles to achieve semistability, by analogy to Hilbert's 21st problem (also known as the Riemann-Hilbert problem).

Hilbert's 21st problem, as answered by Bolibruch [Bol95] (correcting earlier work of Plemelj), poses the question of whether every monodromy representation can be realized by a Fuchsian system. Esnault and Viehweg generalize this question to higher genus in [EV99]: they ask when an irreducible representation can be realized as the monodromy of a flat vector bundle (E, ∇) with regular singularities at infinity, with E semistable.

In Esnault-Viehweg's formulation, the complex structure on the underlying curve is fixed, and the residues of the differential equation at regular singular points are modified to achieve semistability. Flipping this around, Biswas-Heu-Hurtubise's analogue asks if semistability can be achieved by modifying the complex structure and fixing the residues. They claim that this is always possible in the logarithmic, parabolic, and irregular settings, in [BHH16, BHH20, BHH21]. After discovering the Hodge-theoretic counterexample to these claims in Corollary 1.3.3, we proved Theorem 1.3.4 as an attempt (1) to understand to what extent Biswas, Heu, and Hurtubise's Question 1.3.1 has a positive answer, and (2) to apply the cases when there is a positive answer to the analysis of variations of Hodge structure on curves.

1.5. **Idea of proof.** To prove Theorem 1.2.4, we first reduce to proving Theorem 1.2.8, using that discrete compact spaces are finite. We then prove Theorem 1.2.8 by showing that any flat vector bundle satisfying the hypotheses of the theorem is forced to be semistable on an analytically general curve, whence the Hodge filtration consists of a single piece by Corollary 3.1.9. The polarization then gives a definite Hermitian form preserved by the monodromy, and hence the monodromy is unitary. The key issue, which follows from Theorem 1.3.4, is therefore to show that low rank flat vector bundles underlying polarizable complex variations of Hodge structure are semistable on an analytically general curve.

To prove Theorem 1.3.4, we assume we have a flat vector bundle (E, ∇) on our hyperbolic curve (C, D), and consider an isomonodromic deformation to a nearby curve. To this end, we use the deformation theory of this flat vector bundle with its Harder-Narasimhan filtration, which is governed by a variant of the Atiyah bundle. We show that if the Harder-Narasimhan filtration does not satisfy the conclusion of Theorem 1.3.4, then there is a direction we can deform the curve along so that the filtration is destroyed. Indeed, if the filtration persisted, deformation theory provides us with a map from $T_C(-D)$ to a certain semistable subquotient of $\mathrm{End}(E)$ which vanishes on H^1. Taking the Serre duals gives a semistable vector bundle of low rank and slope above $2g-2$ which is not generically globally generated. In the end, we rule this out by a variant of Clifford's theorem for vector bundles.

1.6. **Organization of the paper.** In §2, we give background on Atiyah bundles and isomonodromic deformations. In §3 we give background on complex variations of Hodge structures and their associated Higgs bundles. Experts can likely skip these two sections. In §4, we prove Theorem 1.3.2 and Corollary 1.3.3, providing counterexamples to earlier published claims about semistability of isomonodromic deformations. In § 5 we prove the main results on isomonodromic deformations, Theorem 1.3.4 and Corollary 1.3.6, and is the technical heart of the paper. In § 6, we prove the main consequences for variations of Hodge structure, Theorem 1.2.4, Corollary 1.2.6, and Theorem 1.2.8. Finally, §7 lists some questions motivated by our results.

1.7. **Acknowledgments.** To be added after the referee process is complete.

2. Background on Atiyah bundles and isomonodromic deformations

2.1. **The Atiyah bundle of a filtered vector bundle.** We begin by defining the Atiyah bundle. Let C be a smooth projective curve. Following [Gro65, 16.8.1], for E a vector bundle on C, define $\mathrm{Diff}^1(E, E)$ as follows: for $U \subset C$ open, $\mathrm{Diff}^1(E, E)(U)$ is the set of $\mathbb{C}$-linear endomorphisms τ of $E(U)$, such that for each $f \in \mathscr{O}_C(U), v \in E(U)$, we have that

$$\tau_f : v \mapsto \tau(fv) - f\tau(v)$$

is $\mathscr{O}_C$-linear. Here τ_f measures the failure of τ to be $\mathscr{O}_C$-linear, in that τ_f is zero for all f if and only if τ is $\mathscr{O}_C$-linear.

Definition 2.1.1 (The Atiyah bundle, see [BHH17, p. 5])**.** Let E be a vector bundle on a curve C. Define the Atiyah bundle

$$\mathrm{At}_C(E) \subset \mathrm{Diff}^1(E, E)$$

as the subsheaf with sections on an open set $U \subset C$ given as follows. Let $\mathrm{At}_C(E)(U)$ consist of those $\mathbb{C}$-endomorphisms $\tau \in \mathrm{Diff}^1(E, E)(U)$ such that for each $f \in \mathscr{O}_C(U)$, the endomorphism of E defined by

$$\tau_f : v \mapsto \tau(fv) - f\tau(v)$$

is multiplication by a section $\delta_\tau(f) \in \mathscr{O}_C(U)$.

One can also construct Atiyah bundles associated to filtered bundles.

Definition 2.1.2 (Atiyah bundle of a filtered vector bundle)**.** Let $P^\bullet := (0 = P^0 \subset P^1 \subset \cdots \subset P^n = E)$ be a filtration on E. We define

$$\mathrm{At}_C(E, P^\bullet) \subset \mathrm{At}_C(E)$$

to be the subsheaf consisting of those endomorphisms that preserve $P^\bullet$.

Remark 2.1.3. From the definition, (see also [BHH16, (2.7)]), there is a short exact sequence

$$0 \to \mathrm{End}_{P^\bullet}(E) \xrightarrow{\iota} \mathrm{At}_C(E, P^\bullet) \xrightarrow{\delta} T_C \to 0, \tag{2.1}$$

where $\mathrm{End}_{P^\bullet}(E) \subset \mathrm{End}(E)$ is the subsheaf of $\mathscr{O}_C$-linear endomorphisms preserving $P^\bullet$, ι is the evident inclusion, and δ sends a differential operator τ to the derivation

$$\delta_\tau : f \mapsto \delta_\tau(f)$$

defined in the Definition 2.1.1.

Remark 2.1.4. There is an alternate, perhaps more geometric, description of $\mathrm{At}_C(E, P^\bullet)$. Namely, the filtration $P^\bullet$ gives a restriction of the structure group of E to a parabolic subgroup $P \subset \mathrm{GL}_n$, and hence gives rise to a natural P-torsor $p : \Pi \to C$ over C, which is a subscheme of the frame bundle of E (i.e. it consists of those frames which are compatible with $P^\bullet$). The tangent exact sequence

$$0 \to T_{\Pi/C} \to T_\Pi \to p^* T_C \to 0$$

naturally admits a P-linearization (for the P-action on Π) and hence descends to a short exact sequence on C, which is precisely (2.1).

Next, we introduce Atiyah bundles with respect to divisors.

Definition 2.1.5 (Atiyah bundle of a filtered vector bundle with respect to a divisor)**.** Let $D \subset C$ be a reduced effective divisor. The Atiyah bundle $\mathrm{At}_{(C,D)}(E, P^\bullet)$ is defined as the preimage

$$\mathrm{At}_{(C,D)}(E, P^\bullet) := \delta^{-1}(T_C(-D)),$$

where δ is the map appearing in Sequence (2.1) and where $T_C(-D) \hookrightarrow T_C$ is the natural inclusion.

If $P^\bullet = (0 = P^0 \subset P^1 = E)$ is the trivial filtration, we omit it from the notation, e.g. we will use the notation $\mathrm{At}_{(C,D)}(E)$ in place of $\mathrm{At}_{(C,D)}(E, 0 \subset E)$ when convenient.

Remark 2.1.6. Using Definition 2.1.5 and (2.1), we find that $\mathrm{At}_{(C,D)}(E, P^\bullet)$ fits into a short exact sequence

$$0 \to \mathrm{End}_{P^\bullet}(E) \to \mathrm{At}_{(C,D)}(E, P^\bullet) \to T_C(-D) \to 0. \tag{2.2}$$

By comparing (2.2) for a filtration $P^\bullet$ and the trivial filtration, we obtain the short exact sequence

$$0 \longrightarrow \mathrm{At}_{(C,D)}(E, P^\bullet) \longrightarrow \mathrm{At}_{(C,D)}(E) \longrightarrow \mathrm{End}(E)/\,\mathrm{End}_{P^\bullet}(E) \longrightarrow 0. \tag{2.3}$$

We conclude our review of Atiyah bundles by recalling two important results we will employ. The first gives a description of connections in terms of splittings from the tangent bundle to the Atiyah bundle. The second gives a constraint on such splittings when E is irreducible.

Proposition 2.1.7. *There is a natural bijection between splittings of the Atiyah exact sequence* (2.2) *and flat connections on E with regular singularities along D and preserving $P^\bullet$, given by adjointness. That is, given a connection*

$$\nabla : E \to E \otimes \Omega^1_C(\log D)$$

preserving $P^\bullet$, we may by adjointness view ∇ as a map $T_C(-D) \to \mathrm{End}_{\mathbb{C}}(E)$. This map factors through $\mathrm{At}_{(C,D)}(E)$ and yields a splitting of (2.2)*. Moreover, this correspondence between flat connections and splittings is bijective.*

Proof. This is a matter of unwinding definitions; see [Ati57]. □

Proposition 2.1.8. *Let $(E, \nabla : E \to E \otimes \Omega^1_C(\log D))$ be a flat vector bundle on C with regular singularities along D. Suppose the monodromy representation ρ associated to (E, ∇) via the Riemann-Hilbert correspondence is irreducible. Let $q^\nabla : T_C(-D) \to \mathrm{At}_{(C,D)}(E)$ be the corresponding splitting of the Atiyah exact sequence via Proposition 2.1.7. Then for any nontrivial filtration $P^\bullet$ of E, the composition*

$$T_C(-D) \xrightarrow{q^\nabla} \mathrm{At}_{(C,D)}(E) \to \mathrm{At}_{(C,D)}(E)/\,\mathrm{At}_{(C,D)}(E, P^\bullet) \simeq \mathrm{End}(E)/\,\mathrm{End}_{P^\bullet}(E)$$

is nonzero.

Proof. Assume not. Then q^∇ has image in $\mathrm{At}_{(C,D)}(E, P^\bullet)$, and hence yields a splitting of (2.2). Using Proposition 2.1.7, the corresponding connection preserves P^1 and hence yields a flat connection on P^1 with regular singularities along D, whose monodromy is a sub-representation of the monodromy representation ρ associated to (E, ∇). But this contradicts the assumption that ρ is irreducible. (See [BHH16, Proof of Proposition 5.3] for a similar argument.) $\square$

2.2. **Isomonodromic deformations.** We next recall the notion of isomonodromic deformation. We also define the notion of an "isomonodromic deformation to an analytically general nearby curve" which appears in many of our main results.

Notation 2.2.1. Let $\mathscr{C}, \Delta$ be complex manifolds, and let $\pi : \mathscr{C} \to \Delta$ be a proper holomorphic submersion with connected fibers of relative dimension one, with Δ contractible. Let

$$s_1, \cdots, s_n : \Delta \to \mathscr{C}$$

be disjoint sections to π, and let $\mathscr{D}$ be the union

$$\mathscr{D} := \bigcup_i \mathrm{im}(s_i).$$

Given a point $0 \in \Delta$, let $(C, D) := (\pi^{-1}(0), C \cap \mathscr{D})$ and further assume (C, D) is hyperbolic.

Lemma 2.2.2. *With notation as in Notation 2.2.1, let*

$$(E, \nabla : E \to E \otimes \Omega^1_C(\log D))$$

be a flat vector bundle on C with regular singularities along D. Such a logarithmic flat vector bundle extends canonically to a logarithmic flat vector bundle

$$(\mathscr{E}, \widetilde{\nabla} : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_{\mathscr{C}}(\log \mathscr{D}))$$

on $\mathscr{C}$ with regular singularities along $\mathscr{D}$.

Proof. This follows from Deligne's work on differential equations with regular singularities [Del70] and is explained in [Heu10, Theorem 3.4], following work of Malgrange [Mal83a, Mal83b]. In particular, [Mal83a, Theoreme 2.1] explains the case where C has genus zero, and the general case is similar. We now recapitulate the proof.

The restriction of (E, ∇) to $C \setminus D$ is a flat vector bundle and hence gives rise to a locally constant sheaf of $\mathbb{C}$-vector spaces

$$\mathbb{V} := \ker(\nabla)$$

on $C \setminus D$. As Δ is contractible, the inclusion

$$C \setminus D \hookrightarrow \mathscr{C} \setminus \mathscr{D}$$

is a homotopy equivalence; thus $\mathbb{V}$ extends uniquely (up to canonical isomorphism) to a local system $\widetilde{\mathbb{V}}$ on $\mathscr{C} \setminus \mathscr{D}$.

Manin's local results on extending flat vector bundles across divisors [Del70, Proposition 5.4] imply that there is a canonical extension, which is unique, up to unique isomorphism,

$$(\widetilde{\nabla} : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_{\mathscr{C}}(\log \mathscr{D}))$$

of

$$(\widetilde{\mathbb{V}} \otimes_{\mathbb{C}} \mathscr{O}_{\mathscr{C} \setminus \mathscr{D}}, \mathrm{id} \otimes d)$$

to a flat vector bundle on $\mathscr{C}$ with regular singularities along $\mathscr{D}$, equipped with an isomorphism $(\mathscr{E}, \widetilde{\nabla})|_C \simeq (E, \nabla)$. □

Using the above, we are ready to define isomonodromic deformations.

Definition 2.2.3 (Isomonodromic Deformation)**.** With notation as in Notation 2.2.1, let $D = x_1 + \cdots + x_n$, so that (C, D) is an n-pointed hyperbolic curve of genus g. Let (E, ∇) be a flat vector bundle on C with regular singularities at the x_i. We call the extension $(\mathscr{E}, \widetilde{\nabla})$ as in Lemma 2.2.2 *the isomonodromic deformation* of (E, ∇). If $\Delta = \mathscr{T}_{g,n}$ is the universal cover of of the analytic stack $\mathscr{M}_{g,n}$, and $\mathscr{C} \to \Delta$ is the universal curve, we call the isomonodromic deformation over such Δ *the universal isomonodromic deformation*.

Definition 2.2.4. With notation as in Definition 2.2.3, let Δ be the universal cover of $\mathscr{M}_{g,n}$. We use *an isomonodromic deformation to a nearby curve* to denote the restriction of $(\mathscr{E}, \widetilde{\nabla})$ to any fiber of $\mathscr{C} \to \Delta$. We use *an isomonodromic deformation to an analytically general nearby curve* to denote the restriction of $(\mathscr{E}, \widetilde{\nabla})$ to a general fiber of $\mathscr{C} \to \Delta$, i.e., a fiber in the complement of a nowhere dense closed analytic subset.

Remark 2.2.5. The construction of Lemma 2.2.2 is functorial: given a commutative diagram

$$\begin{array}{ccccc}
D & \hookrightarrow & \mathscr{D} & \longrightarrow & \mathscr{D}' \\
\downarrow & & \downarrow & & \downarrow \\
C & \hookrightarrow & \mathscr{C} & \longrightarrow & \mathscr{C}' \\
\downarrow & & \downarrow \pi & & \downarrow \pi' \\
0 & \hookrightarrow & \Delta & \longrightarrow & \Delta'
\end{array}$$

and a flat vector bundle (E, ∇) on C with regular singularities along D, the isomonodromic deformation over Δ' pulls back to the isomonodromic deformation over Δ.

Example 2.2.6 (Families of families, essentially in [Dor01])**.** With notation as in Notation 2.2.1, suppose $\mathscr{D} = \emptyset$, and let $\tilde{h} : \mathscr{X} \to \mathscr{C}$ be a proper holomorphic submersion. Let $X = h^{-1}(C)$, and let $h = \tilde{h}|_X$. Then for each $i \geq 0$, $R^i\tilde{h}_*\Omega^\bullet_{dR,\mathscr{X}/\mathscr{C}}$ with its Gauss-Manin connection is the isomonodromic deformation of $R^i h_*\Omega^\bullet_{dR,X/C}$ with its Gauss-Manin connection.

2.3. **Deformation theory of isomonodromic deformations.** We now analyze the infinitesimal deformation theory of isomonodromic deformations.

Notation 2.3.1. Let $\mathrm{Art}_{\mathbb{C}}$ be the category of local Artin $\mathbb{C}$-algebras. Let C be a smooth proper curve over $\mathbb{C}$, $D \subset C$ a reduced effective divisor, and

$$(E, \nabla : E \to E \otimes \Omega^1_C(\log D))$$

a flat vector bundle on C with logarithmic singularities along D. Let $P^\bullet$ be a filtration of E.

Definition 2.3.2 (Deformations of a curve with divisor)**.** Let

$$\mathrm{Def}_{(C,D)} : \mathrm{Art}_{\mathbb{C}} \to \mathrm{Set}$$

be the functor sending a local Artin $\mathbb{C}$-algebra $(A, \mathfrak{m}, \kappa)$ (so $\mathfrak{m}$ is the maximal ideal and κ is the residue field) to the set of flat deformations of (C, D) over A. More precisely, it assigns to A the set of those $(\mathscr{C}, \mathscr{D}, q, f)$ where $q : \mathscr{C} \to \operatorname{Spec} A$ is a flat morphism, $\mathscr{D} \subset \mathscr{C}$ is a relative Cartier divisor over $\operatorname{Spec} A$ and $f : C \to \mathscr{C}$ is a map inducing an isomorphism $C \to \mathscr{C} \times_{\operatorname{Spec} A} \operatorname{Spec} \kappa$ taking D isomorphically to $\mathscr{D} \times_{\operatorname{Spec} A} \operatorname{Spec} \kappa$.

Proposition 2.3.3. *With notation as in Definition 2.3.2, there is a canonical and functorial bijection*

$$\mathrm{Def}_{(C,D)}(\mathbb{C}[\epsilon]/\epsilon^2) \xrightarrow{\sim} H^1(C, T_C(-D)).$$

Proof. This is standard, see [Ser06, Proposition 3.4.17]. □

We next generalize the above to describe the deformation theory of filtered vector bundles on curves.

Definition 2.3.4 (Deformations of a filtered vector bundle)**.** Let

$$\mathrm{Def}_{(C,D,E,P^\bullet)} : \mathrm{Art}_{\mathbb{C}} \to \mathrm{Set}$$

be the functor sending A to the set of flat deformations of (C, D, E) over A. More precisely, it assigns to A the set of $(\mathscr{C}, \mathscr{D}, q, f, \mathscr{E}, \mathscr{P}^\bullet, \psi)$ where

$(\mathscr{C}, \mathscr{D}, q, f)$ is a flat deformation of (C, D) over A as in Definition 2.3.2, $\mathscr{E}$ is a vector bundle on $\mathscr{C}$, $\mathscr{P}^\bullet$ is a filtration of $\mathscr{E}$ by sub-bundles, and $\psi : f^*(\mathscr{E}, \mathscr{P}^\bullet) \to (E, P)$ is an isomorphism of filtered vector bundles on C.

Proposition 2.3.5. *Let $(E, P^\bullet)$ be a filtered vector bundle on a curve C. Let $D \subset C$ be a reduced effective divisor. There is a canonical and functorial bijection*

$$\mathrm{Def}_{(C,D,E,P^\bullet)}(\mathbb{C}[\epsilon]/\epsilon^2) \xrightarrow{\sim} H^1(C, \mathrm{At}_{(C,D)}(E, P^\bullet)).$$

Proof. This is explained in [BHH16, §2.2]. □

Remark 2.3.6. If $P^\bullet$ is trivial, we omit it from the notation. In particular,

$$\mathrm{Def}_{(C,D,E)}(\mathbb{C}[\epsilon]/\epsilon^2) \xrightarrow{\sim} H^1(C, \mathrm{At}_{(C,D)}(E)).$$

There is an evident natural transformation

$$\mathrm{Forget} : \mathrm{Def}_{(C,D,E)} \to \mathrm{Def}_{(C,D)}$$

given by forgetting E. The construction of isomonodromic deformations yields a section

$$\mathrm{iso} : \mathrm{Def}_{(C,D)} \to \mathrm{Def}_{(C,D,E)}$$

to this map (which depends on ∇), as we now spell out.

Proposition 2.3.7. *Let $p : \mathrm{At}_{(C,D)}(E) \to T_C(-D)$ be the natural quotient map, and*

$$q^\nabla : T_C(-D) \to \mathrm{At}_{(C,D)}(E)$$

be the section to p arising from ∇ via Proposition 2.1.7. Under the natural identifications

$$\mathrm{Def}_{(C,D)}(\mathbb{C}[\epsilon]/\epsilon^2) \xrightarrow{\sim} H^1(C, T_C(-D))$$

and

$$\mathrm{Def}_{(C,D,E)}(\mathbb{C}[\epsilon]/\epsilon^2) \xrightarrow{\sim} H^1(C, \mathrm{At}_{(C,D)}(E))$$

arising from Proposition 2.3.3 and Proposition 2.3.5, the two squares below commute:

$$\begin{array}{ccc} \mathrm{Def}_{(C,D,E)}(\mathbb{C}[\epsilon]/\epsilon^2) & \xrightarrow{\sim} & H^1(C, \mathrm{At}_{(C,D)}(E)) \\ \downarrow{\scriptstyle \mathrm{Forget}} & & \downarrow{\scriptstyle p_*} \\ \mathrm{Def}_{(C,D)}(\mathbb{C}[\epsilon]/\epsilon^2) & \xrightarrow{\sim} & H^1(C, T_C(-D)) \end{array} \qquad \begin{array}{ccc} \mathrm{Def}_{(C,D,E)}(\mathbb{C}[\epsilon]/\epsilon^2) & \xrightarrow{\sim} & H^1(C, \mathrm{At}_{(C,D)}(E)) \\ \uparrow{\scriptstyle \mathrm{iso}} & & \uparrow{\scriptstyle q^\nabla_*} \\ \mathrm{Def}_{(C,D)}(\mathbb{C}[\epsilon]/\epsilon^2) & \xrightarrow{\sim} & H^1(C, T_C(-D)). \end{array}$$

Proof. This is explained in [BHH16, §2.2 and §4.1]. □

We recall one additional result describing when a filtration extends to a deformation.

Lemma 2.3.8. *Suppose we are given (C, D, E, ∇) and a filtration $P^\bullet$ of E as in Notation 2.3.1. Assume further we have a first-order deformation $(\mathscr{C}, \mathscr{D})$ of (C, D) corresponding to an element $s \in \mathrm{Def}_{(C,D)}(\mathbb{C}[\epsilon]/\epsilon^2) \xrightarrow{\sim} H^1(C, T_C(-D))$. With q^∇ as in Proposition 2.3.7, suppose $q^\nabla(s)$ corresponds to a deformation $(\mathscr{C}, \mathscr{D}, \mathscr{E})$ of (C, D, E) in which $P^\bullet \subset E$ admits an extension to a filtration $\mathscr{P}^\bullet$ of $\mathscr{E}$. Then*

$$q^\nabla(s) \in \ker\left(H^1(C, \mathrm{At}_{(C,D)}(E)) \to H^1(C, \mathrm{End}(E)/\mathrm{End}_{P^\bullet}(E))\right).$$

Proof. Note that the map

$$H^1(C, \mathrm{At}_{(C,D)}(E)) \to H^1(C, \mathrm{End}(E)/\mathrm{End}_{P^\bullet}(E))$$

is induced by the surjection of sheaves $\mathrm{At}_{(C,D)}(E) \to \mathrm{End}(E)/\mathrm{End}_{P^\bullet}(E)$ from (2.3). The proof is explained following the proof of [BHH16, Lemma 3.1].

We now briefly recall the main idea. In the above situation, $q^\nabla(s) \in H^1(C, \mathrm{At}_{(C,D)}(E))$ is in the image of the natural map

$$H^1(C, \mathrm{At}_{(C,D)}(E, P^\bullet)) \to H^1(C, \mathrm{At}_{(C,D)}(E))$$

and the composition

$$H^1(C, \mathrm{At}_{(C,D)}(E, P^\bullet)) \to H^1(C, \mathrm{At}_{(C,D)}(E)) \to H^1(C, \mathrm{End}(E)/\mathrm{End}_{P^\bullet}(E))$$

vanishes. Indeed, this composition is part of the long exact sequence in cohomology induced by the short exact sequence of sheaves (2.3). □

3. Hodge-theoretic preliminaries

We briefly recall the definition of a variation of Hodge structure, and some standard positivity and semistability results for the vector bundles associated to such variations. In particular, Lemma 3.1.6, which shows the first filtered piece of the Hodge filtration tends to have positive degree, is crucial to our semistability arguments. The properties in Proposition 3.1.5 will also be used repeatedly in this paper

3.1. Complex variations of Hodge structure.

Let X be a smooth irreducible complex variety.

Definition 3.1.1 (Polarizable complex variations of Hodge structure)**.** A *complex variation of Hodge structure* on X is a triple $(V, V^{p,q}, D)$, where V is a C^∞ complex vector bundle on X, $V = \oplus V^{p,q}$ is a direct sum decomposition, and D is a flat connection satisfying Griffiths transversality:

$$D(V^{p,q}) \subset A^{1,0}(V^{p,q}) \oplus A^{0,1}(V^{p,q}) \oplus A^{1,0}(V^{p-1,q+1}) \oplus A^{0,1}(V^{p+1,q-1}).$$

A *polarization* on $(V, V^{p,q}, D)$ is a flat Hermitian form ψ on V such that the $V^{p,q}$ are orthogonal to one another under ψ, and such that $(-1)^p\psi$ is positive definite on each $V^{p,q}$. A *polarizable complex variation of Hodge structure* is a complex variation of Hodge structure which admits a polarization.

We call the holomorphic flat vector bundle $(E, \nabla) := (\ker(D) \otimes_{\mathbb{C}} \mathscr{O}, \mathrm{id} \otimes d)$ the *holomorphic flat vector bundle associated to the complex variation of Hodge structure*. The filtration $F^pV := \oplus_{j \geq p} V^{p,q}$ of V induces a decreasing *Hodge filtration* $F^\bullet V$ by holomorphic sub-bundles, such that

$$\nabla(F^p) \subset F^{p-1} \otimes \Omega^1_X. \tag{3.1}$$

If $\mathbb{V}$ is a complex local system on X which is isomorphic to $\ker(D)$ for some polarizable complex variation of Hodge structure $(V, V^{p,q}, D)$, we say that $\mathbb{V}$ *underlies a polarizable complex variation of Hodge structure*.

Definition 3.1.2. Let $\overline{X}$ be a smooth projective variety containing X as a dense open subvariety with simple normal crossings complement Z. We say that a flat holomorphic vector bundle (E, ∇) on X *has (quasi-)unipotent monodromy at infinity* if the monodromy of (E, ∇) about each component of Z is (quasi-)unipotent.

Definition 3.1.3 (Deligne canonical extension [Del70])**.** With notation as in Definition 3.1.2, let (E, ∇) be a flat holomorphic vector bundle on X with unipotent monodromy at infinity. The *Deligne canonical extension* $(\overline{E}, \overline{\nabla} : \overline{E} \to \overline{E} \otimes \Omega^1_{\overline{X}}(\log Z))$ of (E, ∇) to $\overline{X}$ is the unique flat vector bundle on $\overline{X}$ with regular singularities along Z, equipped with an isomorphism $(\overline{E}, \overline{\nabla})|_X \xrightarrow{\sim} (E, \nabla)$, characterized by the property that its residues along Z are nilpotent.

Definition 3.1.4 (The associated Higgs bundle)**.** Let $(E, F^\bullet, \nabla)$ be a holomorphic vector bundle E on a smooth variety $\overline{X}$, with a flat connection ∇ with regular singularities along a simple normal crossings divisor $Z \subset \overline{X}$, and a decreasing filtration $F^\bullet$ by holomorphic sub-bundles satisfying the Griffiths transversality condition

$$\nabla(F^p) \subset F^{p-1} \otimes \Omega^1_{\overline{X}}(\log Z). \tag{3.2}$$

The *associated Higgs bundle* is the pair $(\oplus_i \mathrm{gr}^i_{F^\bullet} E, \theta)$, where the *Higgs field*

$$\theta := \bigoplus_i (\theta_i : \mathrm{gr}^i_{F^\bullet} E \to \mathrm{gr}^{i-1}_{F^\bullet} E \otimes \Omega^1_{\overline{X}}(\log Z))$$

is the $\mathscr{O}_{\overline{X}}$-linear map induced by ∇.

We collect some basic facts about polarizable complex variations of Hodge structure, the canonical extensions thereof, and their associated Higgs bundles:

Proposition 3.1.5. *Let $\overline{X}$ be a smooth projective complex variety, $Z \subset \overline{X}$ a simple normal crossings divisor, and let $X = \overline{X} \setminus Z$. Let L be an ample line bundle on $\overline{X}$. Let $(V, V^{p,q}, D)$ be a polarizable complex variation of Hodge structure on X with unipotent monodromy about the components of Z, $(E, F^\bullet, \nabla)$ be the holomorphic flat vector bundle associated to this variation of Hodge structure, with its Hodge filtration, and let $(\overline{E}, \overline{\nabla})$ be its Deligne canonical extension.*

(1) The local system $\mathbb{V} := \ker(\nabla)$ associated to (E, ∇) is semisimple.

(2) The local system $\mathbb{V}$ may be canonically decomposed as

$$\mathbb{V} \simeq \bigoplus_i \mathbb{L}_i \otimes W_i,$$

where the $\mathbb{L}_i$ are pairwise non-isomorphic irreducible complex local systems on X, and each W_i is a complex vector space. Each $\mathbb{L}_i$ underlies a polarizable complex variation of Hodge structure, and each W_i carries a complex polarized Hodge structure, both unique up to shifting the grading, and compatible with the variation carried by $\mathbb{V}$.

(3) $\overline{E}$ has vanishing rational Chern classes.

(4) There exists a canonical extension of $F^\bullet$ to $\overline{E}$, such that $(\overline{E}, F^\bullet, \overline{\nabla})$ satisfies the Griffiths transversality condition (3.2).

(5) The Higgs bundle $(\oplus_i \operatorname{gr}^i_{F^\bullet} \overline{E}, \theta)$ associated to $(\overline{E}, F^\bullet, \overline{\nabla})$ is polystable of degree zero. That is, set $\deg(H) = c_1(H) \cdot c_1(L)^{\dim \overline{X} - 1}$ for a coherent sheaf H on $\overline{X}$. Then, there exist a collection of vector bundles E_j with $\deg(E_j) = 0$ and maps $\theta_j : E_j \to E_j \otimes \Omega^1_{\overline{X}}(\log Z)$ so that so that $(\oplus_i \operatorname{gr}^i_{F^\bullet} \overline{E}, \theta) = \oplus_j (E_j, \theta_j)$, such that for any θ_j-stable proper subsheaf $H \subset E_j$, $\deg H < 0$.

Proof. The proof of (1) is explained in [Del87, 1.11-1.12] and (2) is [Del87, 1.13]. The proof of (3) follows from [EV86, B.3] while (4) follows from the Nilpotent Orbit Theorem [CKS86, 2.2(1)]. Finally, (5) is explained on [Ara19, p. 4]; the case where $\overline{X}$ is a curve, which is all we will use in this paper, is due to Simpson [Sim90, Theorem 5]. See the discussion in the introduction of [Ara19] for a nice summary of this and related topics. □

The next lemma is crucial in the proof of our main result Theorem 1.2.8 since the positivity it gives for $F^i\overline{E}$ will contradict our later results on semistability, unless the Hodge filtration has only a single part. The connection to semistability is spelled out below in Corollary 3.1.9.

Lemma 3.1.6. *Let C be a smooth proper curve, $Z \subset C$ a reduced effective divisor, and $(V, V^{p,q}, D)$ a a polarizable complex variation of Hodge structure on $C \setminus Z$ with unipotent monodromy around the components of Z. Let $(\overline{E}, F^\bullet, \nabla)$ be the Deligne canonical extension of the associated flat holomorphic vector bundle to C. Let i be* *maximal such that $F^i\overline{E}$ is non-trivial, where $F^\bullet$ is the Hodge filtration, and suppose that the Higgs field*

$$\theta_i : F^i\overline{E} \to \mathrm{gr}^{i-1}_{F^\bullet}\,\overline{E} \otimes \Omega^1_C(\log Z)$$

is non-zero. Then $F^i\overline{E}$ has positive degree.

Remark 3.1.7. Lemma 3.1.6 is essentially due to Griffiths. In the case of real variations (as opposed to complex variations) of Hodge structure on a smooth proper curve, it follows from the curvature formula [Gri70, Theorem 5.2], and is observed there in some special cases [Gri70, Corollary 7.10]. For a more precise reference, see [Pet00, Corollary 2.2], which immediately implies the claim for real variations of Hodge structure on a smooth proper curve. However, as we were unable to find a reference in the case of complex variations, we now give a simple proof.

Proof of Lemma 3.1.6. The vector bundle $F^i\overline{E}$ with the zero Higgs field is a quotient of the Higgs bundle $(\oplus_i \mathrm{gr}^i_{F^\bullet}\,\overline{E}, \theta)$, and hence has non-negative degree by the fact that the latter is polystable of degree zero, by Proposition 3.1.5(5). It has degree zero if and only if it is a direct summand of $(\oplus_i \mathrm{gr}^i_{F^\bullet}\,\overline{E}, \theta)$ by polystability; but this is ruled out by the nonvanishing of θ_i. ☐

Remark 3.1.8. By the construction of the Higgs field θ, the condition that θ_i is non-zero in Lemma 3.1.6 is equivalent to the statement that $F^i\overline{E}$ is not preserved by $\overline{\nabla}$. For example, it is automatically nonzero if $(\overline{E}, \overline{\nabla})$ has irreducible monodromy and $F^i\overline{E}$ is a proper sub-bundle of $\overline{E}$.

We now spell out how Lemma 3.1.6 relates to semistability.

Corollary 3.1.9. *Let $(\overline{E}, F^\bullet, \nabla)$ be as in Lemma 3.1.6. Then the vector bundle $\overline{E}$ is not semistable.*

Proof. The vector bundle $\overline{E}$ has degree zero by Proposition 3.1.5(3). But by Lemma 3.1.6, $F^i\overline{E}$ has positive degree, and is hence a destabilizing subsheaf. ☐

4. Variations of Hodge structure and the Kodaira-Parshin trick

In this section we find that variations of Hodge structure on $\mathscr{M}_{g,1}$ with monodromy which is "big" in a suitable sense provide examples of flat vector bundles on curves whose isomonodromic deformation to a nearby curve is never semistable. We then produce such variations of Hodge structure via the Kodaira-Parshin trick. This will be used to prove Theorem 1.3.2 and contradicts earlier claimed theorems of Biswas, Heu, and Hurtubise [BHH16, Theorem 1.3], [BHH21, Theorem 1.3], and [BHH20, Theorem 1.2], as described further in Remark 4.1.9.

In §6, we will use that variations of Hodge structure with suitably large monodromy yield flat vector bundles which do not have isomonodromic deformations to semistable bundles. This will be used to analyze variations of Hodge structure on an analytically very general curve.

4.1. **Construction of the counterexample.** We now set up the proof of Theorem 1.3.2 and Corollary 1.3.3. We will construct a variation of Hodge structure over the analytic moduli stack $\mathscr{M}_{g,1}$ whose restriction to each fiber of the forgetful map $\mathscr{M}_{g,1} \to \mathscr{M}_g$ satisfies the hypotheses of Lemma 3.1.6. We will do this via the Kodaira-Parshin trick (see [Par68, Proposition 7] and also [LV20, Proposition 7.1]), which produces a family of curves over $\mathscr{M}_{g,1}$ which is non-isotrivial when restricted to each fiber of the natural forgetful map $\mathscr{M}_{g,1} \to \mathscr{M}_g$. We give a proof appealing to [CD17], but one can also prove it using Proposition 3.1.5 and Corollary 3.1.9, as we mention in Remark 4.1.8 Because we had difficulty finding a suitable reference, we now present a version of the Kodaira-Parshin trick in families.

Proposition 4.1.1 (Kodaira-Parshin trick)**.** *Let Y denote a Riemann surface of genus $g \geq 1$ with a point $p \in Y$ and let $Y^\circ := Y - p$. Choose a basepoint $y \in Y^\circ$. Suppose G is a finite center-free group with a surjection $\phi : \pi_1(Y^\circ, y) \twoheadrightarrow G$ which sends the loop around the puncture $p \in Y$ to a non-identity element of G. Then there exists a smooth proper relative dimension 1 map of analytic stacks $f : \mathscr{X} \to \mathscr{M}_{g,1}$ so that the fiber over a geometric point $[(C, p)] \in \mathscr{M}_{g,1}$ is a finite disjoint union of G-covers of C ramified at p.*

We will prove this below in §4.1.5.

Remark 4.1.2. In the finite disjoint union of G-covers appearing at the end of the statement of Proposition 4.1.1, we can explicitly identify the finite set of G-covers. Namely, suppose $h \in \pi_1(\mathscr{M}_{g,2})$, viewed as an automorphism of the fundamental group of a 2-pointed genus g curve $\pi_1(Y^\circ, y)$. (See §4.1.3 below for an explanation of the action of $\pi_1(\mathscr{M}_{g,2})$ on $\pi_1(Y^\circ, y)$.) There is one cover associated to each map of the form $\phi_h : \pi_1(Y^\circ, y) \to G$, with $\phi_h(g) := \phi(hgh^{-1})$, modulo the following equivalence relation: we identify $\phi_h \sim \phi_g$ if they are conjugate, i.e., if there exists $m \in G$ with $\phi_h(s) = m\phi_g(s)m^{-1}$ for all $s \in \pi_1(Y^\circ, y)$.

4.1.3. *Setup to prove Proposition 4.1.1.* Let

$$\pi : \mathscr{M}_{g,2} \to \mathscr{M}_{g,1}$$

be the natural forgetful map. Let $x \in \mathscr{M}_{g,2}$ be a point. Let $\bar{x} := \pi(x) \in \mathscr{M}_{g,1}$ and let $C^\circ \subset \mathscr{M}_{g,2}$ denote the fiber of $\pi^{-1}(\bar{x})$.

Note that C° is the complement of a point in a smooth proper connected curve of genus g. There is a natural short exact sequence

$$1 \to \pi_1(C^\circ, x) \to \pi_1(\mathscr{M}_{g,2}, x) \to \pi_1(\mathscr{M}_{g,1}, \bar{x}) \to 1 \tag{4.1}$$

associated to the map $\mathscr{M}_{g,2} \to \mathscr{M}_{g,1}$ with fiber C°. We may obtain this from the Birman exact sequence [FM12, Theorem 4.6] for mapping class groups, after identifying the fundamental group for $\mathscr{M}_{g,n}$ with the mapping class group of an n-times punctured genus g surface. (The case $n = 0$ follows from contractibility of the universal cover of $\mathscr{M}_g$ [FM12, Theorem 10.6] with covering group given by the mapping class group, and the case of general n can be deduced from the Birman exact sequence [FM12, Theorem 4.6].)

Let G be a center-free finite group and suppose further there is a surjection

$$\gamma : \pi_1(C^\circ, x) \twoheadrightarrow G.$$

We assume that γ takes the conjugacy class of the loop around the puncture of C° to a non-identity conjugacy class of G.

Define $\Gamma \subset \pi_1(\mathscr{M}_{g,2}, x)$ as the set of $h \in \pi_1(\mathscr{M}_{g,2}, x)$ such that there exists $\widetilde{\gamma}(h) \in G$ with

$$\gamma(hgh^{-1}) = \widetilde{\gamma}(h)\gamma(g)\widetilde{\gamma}(h)^{-1}$$

for all $g \in \pi_1(C^\circ, x)$.

Lemma 4.1.4. *Keeping notation as in §4.1.3, the map γ determines a well-defined surjective homomorphism*

$$\begin{aligned} \widetilde{\gamma} : \Gamma &\to G \\ h &\mapsto \widetilde{\gamma}(h). \end{aligned}$$

Moreover, $\Gamma \subset \pi_1(\mathscr{M}_{g,2}, x)$ has finite index.

Proof. We first claim that Γ contains $\pi_1(C^\circ, x)$ and surjects onto G. Indeed, for $h \in \pi_1(C^\circ, x)$, one may take $\widetilde{\gamma}(h) = \gamma(h)$. Therefore, the surjectivity of γ implies that $\widetilde{\gamma}$ is also surjective.

Next, we claim that for each h, $\widetilde{\gamma}(h)$ is uniquely determined. Indeed, suppose $\widetilde{\gamma}(h)$ may be either α and β. Then we would have $\alpha\gamma(g)\alpha^{-1} = \beta\gamma(g)\beta^{-1}$. Since γ is surjective, as shown above, we find $\alpha\beta^{-1}$ lies in the center of G, and therefore is trivial. So $\alpha = \beta$.

The uniqueness of $\widetilde{\gamma}(h)$ just established shows that $\widetilde{\gamma}$ determines a well-defined map. This is moreover a homomorphism by the above established uniqueness, because we then obtain $\widetilde{\gamma}(h)\widetilde{\gamma}(h') = \widetilde{\gamma}(hh')$.

Finally, we claim Γ has finite index in $\pi_1(\mathscr{M}_{g,2}, x)$. To see this, observe that there is an action of $\pi_1(\mathscr{M}_{g,2}, x)$ on the set of surjective homomorphisms $\pi_1(C^\circ, x) \to G$ sending $\phi : \pi_1(C^\circ, x) \to G$ to the map $\phi^h(g) := \phi(hgh^{-1})$.

By definition, we have $h \in \Gamma$ if and only if γ^h is conjugate to γ. In particular, Γ contains the stabilizer of γ under the action of $\pi_1(\mathscr{M}_{g,2}, x)$. But this stabilizer has finite index in $\pi_1(\mathscr{M}_{g,2}, x)$ because G is finite and $\pi_1(C^\circ, x)$ is finitely generated, so there are only finitely many surjective homomorphisms $\pi_1(C^\circ, x) \to G$. $\square$

4.1.5.

Proof of Proposition 4.1.1. Let $\widetilde{\Gamma}$ be the kernel of the map $\widetilde{\gamma}$ from Lemma 4.1.4. The subgroup $\widetilde{\Gamma}$ corresponds to a finite étale cover $\mathscr{X}^\circ \to \mathscr{M}_{g,2}$. Observe that $\mathscr{M}_{g,2} \subset \mathscr{M}_{g,1} \times_{\mathscr{M}_g} \mathscr{M}_{g,1}$ can be viewed as a dense open substack, and let $\mathscr{X}$ be the normalization of $\mathscr{M}_{g,1} \times_{\mathscr{M}_g} \mathscr{M}_{g,1}$ in the function field of $\mathscr{X}^\circ$, forming the following cartesian diagram

$$\begin{array}{ccc} \mathscr{X}^\circ & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathscr{M}_{g,2} & \longrightarrow & \mathscr{M}_{g,1} \times_{\mathscr{M}_g} \mathscr{M}_{g,1}. \end{array} \tag{4.2}$$

Restricting the natural map $\mathscr{X} \to \mathscr{M}_{g,1} \times_{\mathscr{M}_g} \mathscr{M}_{g,1}$ to a fiber C of the universal curve $\mathscr{M}_{g,1} \times_{\mathscr{M}_g} \mathscr{M}_{g,1} \to \mathscr{M}_{g,1}$ yields a finite disjoint union of G-covers of C, ramified only over the tautological marked point of C. We then take our desired relative curve $f : \mathscr{X} \to \mathscr{M}_{g,1} \times_{\mathscr{M}_g} \mathscr{M}_{g,1} \to \mathscr{M}_{g,1}$ as the resulting composition. $\square$

In order to use Proposition 4.1.1, we will need to know there are groups G satisfying its hypotheses. We now provide such an example.

Example 4.1.6. As a concrete example of a group G to which Proposition 4.1.1 applies, we can take $G = S_3$ to be the symmetric group on three letters and identify $\pi_1(Y^\circ, y)$ with the free group on the generators $a_1, \ldots, a_g, b_1, \ldots, b_g$. The group $\pi_1(Y, y)$ is generated by $a_1, \ldots, a_g, b_1, \ldots, b_g$ with the relation $\prod_{i=1}^g [a_i, b_i]$. Consider the surjection $\phi : \pi_1(C^\circ, y) \twoheadrightarrow S_3$ sending $a_1 \mapsto (12), b_1 \mapsto (13)$ and sending $a_i \mapsto \mathrm{id}, b_i \mapsto \mathrm{id}$ for $i > 1$. The loop around the puncture maps to $\phi(\prod_{i=1}^g [a_i, b_i]) = (12)(13)(12)^{-1}(13)^{-1} = (132) \neq \mathrm{id}$.

4.1.7.

Proof of Theorem 1.3.2. Let $f : \mathscr{X} \to \mathscr{M}_{g,1}$ denote the map from Proposition 4.1.1. Concretely, we can take $G = S_3$ and the map ϕ as in Proposition 4.1.1 to be that given in Example 4.1.6. Define the local system $\mathbb{V} := R^1 f_* \mathbb{C}$ on $\mathscr{M}_{g,1}$, and define $\mathscr{F}$ to be the vector bundle $\mathbb{V} \otimes \mathscr{O}$. Note that $\mathscr{F}$ admits a natural (Gauss-Manin) connection $\mathrm{id} \otimes d$. The local system $\mathbb{V}$ evidently underlies a variation of Hodge structure.

Let C be a fiber of the forgetful morphism $\mathscr{M}_{g,1} \to \mathscr{M}_g$. Let $X := f^{-1}(C) \subset \mathscr{X}$. We claim that the flat vector bundle $(\mathscr{F}, \nabla)$ satisfies the conditions of Theorem 1.3.2, i.e. it has semisimple monodromy and $\mathscr{F}|_C$ is not semistable.

We first check that $(\mathscr{F}, \nabla)|_C$ has semisimple monodromy. This is true for any flat vector bundle arising from the Gauss-Manin connection on the cohomology of a family of smooth proper varieties, by work of Deligne [Del71, Théorème 4.2.6].

We now check that $(\mathscr{F}, \nabla)|_C$ is not semistable. By [CD17, Theorem 4], if $X \to C$ is not isotrivial, $f_*\omega_f$ is a destabilizing subsheaf of $\mathscr{F}$. It remains to show $X \to C$ is not isotrivial. The fiber of $f|_X$ over a point $x \in C$ is a finite disjoint union of finite covers of C, branched only over x. These fibers must vary in moduli as x varies, as there are only finitely many non-constant maps between any two curves over of genus at least 2, by de Franchis' theorem. (See [dF13] or [SA66, Corollary 3, p. 75], for example.) □

Remark 4.1.8. We can also give a somewhat more involved proof of Theorem 1.3.2 using Corollary 3.1.9 in place of [CD17, Theorem 4], as we now explain. This argument inspired the Hodge-theoretic results Theorem 1.2.4 and Corollary 1.2.6, proven in §6.

With notation as in the proof of Theorem 1.3.2, $\mathscr{F}$ has degree 0 since it admits a flat connection, by Proposition 3.1.5(3). Therefore, it suffices to show $\mathscr{F}$ has a subsheaf of positive degree. The Hodge filtration exhibits $F^1\mathscr{F} \simeq (f|_X)_*\omega_{(f|_X)}$ as a subsheaf of $\mathscr{F}$, which is destabilizing by Corollary 3.1.9 once we verify that $\delta : F^1\mathscr{F} \to F^0\mathscr{F}/F^1\mathscr{F} \simeq R^1(f|_X)_*\mathscr{O}_X$ is nonzero.

We now check δ is nonzero. Locally around a point of C, δ can be identified with the derivative of the period map [CMSP17, Theorem 5.3.4] sending a curve corresponding to a fiber of $f|_X : X \to C$ to the corresponding Hodge structure on its first cohomology. To show this derivative is non-zero it suffices to show that the period map is non-constant. More concretely, by the Torelli theorem, we only need to check $f|_X : X \to C$ is not isotrivial. This follows by de Franchis' theorem, as explained in the proof of Theorem 1.3.2.

Remark 4.1.9. As noted prior to the statement of Theorem 1.3.2, Theorem 1.3.2 contradicts [BHH16, Theorem 1.3], [BHH21, Theorem 1.3], and [BHH20, Theorem 1.2], which claim, for example, that any irreducible flat vector bundle on a smooth proper curve of genus at least 2 admits a semistable isomonodromic deformation. We now explain the gaps in the proofs of those results. The error in [BHH16, Theorem 1.3] occurs in [BHH16, Proposition 4.3]; the issue is that the map denoted $f^*\nabla$ in diagram (4.14) does not in general exist. The proof works correctly if $G = \mathrm{GL}_2$. An identical error occurs in [BHH21, Proposition 4.4]. A different argument is given in [BHH20]. There, the error occurs in the proof of [BHH20, Proposition 5.1], in which the large diagram claimed to be commutative does not in general commute.

5. Analysis of Harder-Narasimhan filtration

In this section, we prove Theorem 1.3.4. Recall that a vector bundle V is not generically globally generated if the evaluation map $H^0(C, V) \otimes \mathscr{O}_C \to V$ factors through a proper subbundle of V. The basic idea will be to show that any counterexample to Theorem 1.3.4 will produce a certain semistable vector bundle of high slope which is not generically globally generated. In order to see why this failure of generic global generation leads to a contradiction, we will need some facts about (generic) global generation of vector bundles on curves, arising from Clifford's theorem.

5.1. **Preliminary results on high slope bundles with many sections.**

We start with a bound on the dimension of the space of global sections of a vector bundle whose Harder-Narasimhan polygon has slopes between 0 and $2g$.

Lemma 5.1.1. *Suppose V is a vector bundle on a smooth proper curve C with Harder-Narasimhan filtration $0 = N^0 \subset N^1 \subset \cdots \subset N^n = V$. Suppose moreover that for each i, the slope of $\mathrm{gr}^i_N V = N^i/N^{i-1}$ satisfies*

$$0 \leq \mu(\mathrm{gr}^i_N V) := \frac{\deg(\mathrm{gr}^i_N V)}{\mathrm{rk}(\mathrm{gr}^i_N V)} \leq 2g.$$

Then $\dim H^0(C, V) \leq \frac{\deg V}{2} + \mathrm{rk}\, V$.

Proof. For convenience set $W_i := \mathrm{gr}^i_N V = N^i/N^{i-1}$. Suppose $W_1, \ldots, W_k$ have slopes $> 2g-2$ and $W_{k+1}, \ldots, W_n$ have slopes $\leq 2g-2$.

Using Clifford's theorem for vector bundles [BPGN97, Theorem 2.1], for $i > k$, we have

$$\dim H^0(C, W_i) \leq \frac{\deg W_i}{2} + \mathrm{rk}\, W_i.$$

Also, for $i \leq k$, since W_i are semistable, there are no maps $W_i \to \omega_C$. Therefore, $H^1(C, W_i) = 0$ when $i \leq k$. It follows from Riemann Roch that

$$\dim H^0(C, W_i) = \deg W_i + (1-g)\,\mathrm{rk}\, W_i$$

for $i \leq k$. Summing over i, we get

$$\begin{aligned}
\dim H^0(C, W) &\leq \sum_{i=1}^{n} \dim H^0(C, W_i) \\
&\leq \sum_{i=1}^{k} (\deg W_i + (1-g) \operatorname{rk} W_i) + \sum_{i=k+1}^{n} (\frac{\deg W_i}{2} + \operatorname{rk} W_i) \\
&= \sum_{i=1}^{n} (\frac{\deg W_i}{2} + \operatorname{rk} W_i) + \sum_{i=1}^{k} (\frac{\deg W_i}{2} - g \operatorname{rk} W_i) \\
&= \frac{\deg W}{2} + \operatorname{rk} W + \sum_{i=1}^{k} (\frac{\deg W_i}{2} - g \operatorname{rk} W_i).
\end{aligned}$$

To conclude, it is enough to show $\frac{\deg W_i}{2} - g \operatorname{rk} W_i \leq 0$. However, since we were assuming the slope $\mu(W_i) \leq 2g$, we find $\deg W_i \leq 2g \operatorname{rk} W_i$ and so $\frac{\deg W_i}{2} \leq g \operatorname{rk} W_i$, as desired. □

The following lemma is a well known criterion for global generation, which we spell out for completeness.

Lemma 5.1.2. *Let V be a semistable vector bundle on a smooth proper curve C, such that the slope of V satisfies $\mu(V) > 2g - 1$. Then V is globally generated.*

Proof. Let $p \in C$ be a point. It suffices to show V is globally generated by global sections at p. Indeed, $V(-p)$ is a semistable bundle with slope $\mu(V(-p)) > 2g - 2$. Hence $H^1(C, V(-p)) = 0$, as any map $V(-p) \to \omega_C$ would be destabilizing. Since $H^1(C, V(-p)) = 0$, the sequence

$$0 \longrightarrow V(-p) \longrightarrow V \longrightarrow V|_p \longrightarrow 0 \tag{5.1}$$

is exact on global sections, so $H^0(C, V) \otimes \mathscr{O} \to V \to V|_p$ is surjective, as desired. □

The following result will be key to the proof of Theorem 1.3.4, as it places a constraint on the rank of a vector bundle which is not generically globally generated.

Proposition 5.1.3. *Suppose V is a semistable vector bundle on a smooth proper curve C, with slope $\mu(V) > 2g - 2$. If V is not generically globally generated, then* $\operatorname{rk} V \geq g + 1$.

Proof. We will verify the hypotheses for Lemma 5.1.4. Since V is semistable of slope more than $2g - 2$, V has no maps to ω_C and hence $H^1(C, V) = 0$. Let $U \subset V$ be the saturation of the image of the evaluation map

$$H^0(C, V) \otimes \mathscr{O}_C \to V.$$

Using Lemma 5.1.2, we may assume $2g-2 < \mu(V) \leq 2g-1$. If V is not generically globally generated, $U \subset V$ is a proper sub-bundle of V, with $H^0(C,U) \to H^0(C,V)$ an isomorphism. Since $\mu(V) \leq 2g-1$ and V is semistable, each graded piece $\mathrm{gr}^i_{\mathrm{HN}} U$ of the Harder-Narasimhan filtration of U must have slope at most $2g-1$. Let j be maximal such that $\mathrm{gr}^j_{\mathrm{HN}} U$ is non-zero. Since U is generically globally generated, $\mathrm{gr}^j_{\mathrm{HN}} U$ has a global section, and therefore has non-negative slope. By the definition of the Harder-Narasimhan filtration, the same is true for every graded piece. This verifies the hypotheses of Lemma 5.1.4, so we conclude $\mathrm{rk}\, V \geq g+1$. □

Lemma 5.1.4. *Suppose V is a vector bundle on a smooth proper curve C with $\mu(V) > 2g-2$ and $H^1(C,V) = 0$. Assume further $U \subset V$ is a proper subbundle such that*

(1) $H^0(C,U) = H^0(C,V)$,
(2) $\mu(U) \leq \mu(V)$, *and*
(3) *each graded piece $\mathrm{gr}^i_{\mathrm{HN}} U$ of the Harder Narasimhan filtration of U satisfies $0 \leq \mathrm{gr}^i_{\mathrm{HN}} U \leq 2g$.*

Then, $\mathrm{rk}\, V \geq g+1$.

Proof. Applying Lemma 5.1.1, we conclude

$$H^0(C,U) \leq \frac{\deg U}{2} + \mathrm{rk}\, U.$$

Since $H^1(C,V) = 0$,

$$\dim H^0(C,U) = \dim H^0(C,V) = \deg V + (1-g)\,\mathrm{rk}\, V.$$

Therefore, we get

$$\deg V + (1-g)\,\mathrm{rk}\, V \leq \frac{\deg U}{2} + \mathrm{rk}\, U.$$

Rewriting this, and using $\mathrm{rk}\, U \leq \mathrm{rk}\, V - 1$ and $\mu(U) \leq \mu(V)$ gives

$$\mu(V)\,\mathrm{rk}(V) + (1-g)\,\mathrm{rk}\, V \leq \frac{\mu(U)\,\mathrm{rk}\, U}{2} + \mathrm{rk}\, U \leq \frac{\mu(V)}{2}(\mathrm{rk}\, V - 1) + \mathrm{rk}\, V - 1.$$

Rearranging the terms, and multiplying both sides by 2, we obtain

$$\mu(V) \leq (2g - \mu(V))\,\mathrm{rk}\, V - 2.$$

Since $2g-2 < \mu(V)$, we find $2g - \mu(V) < 2$ and hence

$$2g-2 < \mu(V) \leq (2g-\mu(V))\,\mathrm{rk}\, V - 2 < 2\,\mathrm{rk}\, V - 2.$$

Therefore, $\mathrm{rk}\, V > g$ as desired. □

5.2. **Reduction for the proof of Theorem 1.3.4.** We next prove some preparatory results to complete the proof of Theorem 1.3.4. Reviewing the idea of the proof, described in §1.5, may be helpful.

Notation 5.2.1. Let (C, D) be a hyperbolic curve. Let (E, ∇) be a flat vector bundle on C with regular singularities along D, whose associated monodromy representation is irreducible. Let $N^\bullet$, given by $0 = N^0 \subset N^1 \subset \cdots \subset N^n = E$, be a nontrivial filtration of E, so $n > 1$. Let $\mathrm{gr}^i_N(E) := N^i/N^{i-1}$. The sheaf $\mathrm{End}(E)/\mathrm{End}_{N^\bullet}(E)$ has a filtration by sheaves whose associated graded sheaf is of the form

$$\oplus_{1\le i<j\le n} \mathrm{Hom}(\mathrm{gr}^i_N(E), \mathrm{gr}^j_N(E)).$$

For $i < j$ define $E_{i,j} := \mathrm{Hom}(\mathrm{gr}^i_N(E), \mathrm{gr}^j_N(E))$.

Let $\Delta = \mathscr{T}_{g,n}$ be the universal cover of the analytic stack $\mathscr{M}_{g,n}$, and let $(\mathscr{C}, \mathscr{D})$ be the universal marked curve over $\mathscr{T}_{g,n}$. Let $0 \in \Delta$ be such that $(\mathscr{C}, \mathscr{D})_0$ is isomorphic to (C, D); fix such an isomorphism. Let $(\mathscr{E}, \widetilde{\nabla})$ be the universal isomonodromic deformation of (E, ∇) to $\mathscr{C}$.

We will later take the filtration $N^\bullet$ to be the Harder-Narasimhan filtration of E.

By Proposition 2.1.8, the connection ∇ yields a non-zero map

$$T_C(-D) \xrightarrow{q^\nabla} \mathrm{At}_{(C,D)}(E) \to \mathrm{End}(E)/\mathrm{End}_{N^\bullet}(E). \tag{5.2}$$

We now observe that if $N^\bullet$ extends to the universal isomonodromic deformation of (C, D, E), the induced map on first cohomology must vanish.

Lemma 5.2.2. *Retain notation as in Notation 5.2.1. If the filtration $N^\bullet$ extends to a filtration on the restriction of $(\mathscr{E}, \widetilde{\nabla})$ to a first-order neighborhood of $(C, D) = (\mathscr{C}, \mathscr{D})_0 \subset (\mathscr{C}, \mathscr{D})$, then the composite map*

$$H^1(C, T_C(-D)) \xrightarrow{q^\nabla_*} H^1(C, \mathrm{At}_{(C,D)}(E)) \to H^1(C, \mathrm{End}(E)/\mathrm{End}_{N^\bullet}(E)).$$

induced by (5.2) is identically zero.

Proof. By Proposition 2.3.7 the map

$$q^\nabla_* : H^1(C, T_C(-D)) \to H^1(C, \mathrm{At}_{(C,D)}(E))$$

induced by the connection sends a first-order deformation of the pointed curve (C, D) to the corresponding first-order deformation of the triple (C, D, E) obtained from isomonodromically deforming the connection ∇. But given a first-order deformation $(\widetilde{C}, \widetilde{D}, \widetilde{E})$ of (C, D, E) such that $N^\bullet \subset E$ admits an extension $\widetilde{N}^\bullet$ to $\widetilde{E}$, the corresponding element of $H^1(C, \mathrm{At}_{(C,D)}(E))$ maps to 0 in $H^1(C, \operatorname{End}(E)/\operatorname{End}_{N^\bullet}(E))$, by Lemma 2.3.8. The assumption is precisely that this is true for all elements of $H^1(C, \operatorname{At}_{(C,D)}(E))$ in the image of q_*^∇. $\square$

We now analyze the vector bundles $E_{i,j} := \operatorname{Hom}(\operatorname{gr}_N^i(E), \operatorname{gr}_N^j(E)), i < j$.

Lemma 5.2.3. *With notation as in Notation 5.2.1, for every $0 < i < n$, there exists j, k with $j < i$ and $k \geq i+1$ so that the nonzero map $T_C(-D) \to \operatorname{End}(E)/\operatorname{End}_{N^\bullet}(E)$ induces a nonzero map $\phi_{j+1,k} : T_C(-D) \to E_{j+1,k}$.*

Proof. First, by Proposition 2.1.8, the map $T_C(-D) \to \operatorname{End}(E)/\operatorname{End}_{N^\bullet}(E)$ is nonzero. Let j be maximal such that $\nabla(N^j) \subset N^i \otimes \Omega_C^1(\log D)$. Note that $j < i$ as the monodromy of (E, ∇) is irreducible, so N^i is not a proper flat subbundle of (E, ∇), implying $\nabla(N^i) \not\subset N^i \otimes \Omega_C^1(\log D)$. Let k be minimal such that $\nabla(N^{j+1}) \subset N^k \otimes \Omega^1(D)$. Note that $k \geq i+1$ by the definition of j. By construction, the connection induces a nonzero $\mathscr{O}_C$-linear map

$$N^{j+1}/N^j \to (N^k/N^i) \otimes \Omega_C^1(D) \to (N^k/N^{k-1}) \otimes \Omega_C^1(D),$$

or equivalently a nonzero map

$$\phi_{j+1,k} : T_C(-D) \to \operatorname{Hom}(\operatorname{gr}_N^{j+1}(E), \operatorname{gr}_N^k(E)) = E_{j+1,k}. \quad \square$$

We have shown that for each i, there exist $j < i < k$, and a non-zero map

$$T_C(-D) \to E_{j+1,k} = \operatorname{Hom}(\operatorname{gr}_N^{j+1}(E), \operatorname{gr}_N^k(E)).$$

We next refine Lemma 5.2.2 by showing that if $N^\bullet$ is the Harder-Narasimhan filtration of E, the map on H^1 induced by $\phi_{j+1,k} : T_C(-D) \to E_{j+1,k}$ must also vanish.

Remark 5.2.4. Suppose the filtration $N^\bullet$ appearing in Notation 5.2.1 is the Harder-Narasimhan filtration of E.

Because a tensor product of semistable sheaves is semistable in characteristic 0, the $E_{i,j}$ are semistable in this case. [HL10, Theorem 3.1.4].

Note that as $i < j$, $E_{i,j}$ has negative degree since $\deg \operatorname{gr}_N^i(E) > \deg \operatorname{gr}_N^{i+1}(E)$. by the definition of the Harder-Narasimhan filtration.

Lemma 5.2.5. *Notation as in Notation 5.2.1. Suppose in addition that $N^\bullet$ is the Harder-Narasimhan filtration of E. Fix i with $0 < i < n$ and let j, k, and*

$$\phi_{j+1,k} : T_C(-D) \to E_{j+1,k}$$

be the data constructed in Lemma 5.2.3. If the filtration $N^\bullet$ extends to a filtration on the restriction of $(\mathscr{E}, \widetilde{\nabla})$ to a first-order neighborhood of $(C, D) = (\mathscr{C}, \mathscr{D})_0 \subset (\mathscr{C}, \mathscr{D})$, then the map $H^1(C, T_C(-D)) \to H^1(C, E_{j+1,k})$ induced by $\phi_{j+1,k}$ vanishes.

Proof. The proof is a diagram chase. We first show there is a natural map $T_C(-D) \to \mathrm{Hom}(N^{j+1}, E/N^{k-1})$ which vanishes on H^1. The natural surjection of sheaves $\mathrm{End}(E) \twoheadrightarrow \mathrm{Hom}(N^{j+1}, E)$ induces a surjection

$$\mathrm{End}(E)/\,\mathrm{End}_{N^\bullet}(E) \twoheadrightarrow \mathrm{Hom}(N^{j+1}, E/N^{k-1})$$

and hence a surjection

$$H^1(C, \mathrm{End}(E)/\,\mathrm{End}_{N^\bullet}(E)) \twoheadrightarrow H^1(C, \mathrm{Hom}(N^{j+1}, E/N^{k-1})).$$

Thus the composition $T_C(-D) \to \mathrm{End}(E)/\,\mathrm{End}_{N^\bullet}(E) \to \mathrm{Hom}(N^{j+1}, E/N^{k-1})$ induces the zero map on H^1, by Lemma 5.2.2.

We next show the natural map $T_C(-D) \to \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), E/N^{k-1})$ to be described below, vanishes on H^1. As a first step, we claim that

$$H^0(C, \mathrm{Hom}(N^j, E/N^{k-1})) = 0.$$

This holds because the slopes of the Harder-Narasimhan constituents of $\mathrm{Hom}(N^j, E/N^{k-1})$ are negative, by the definition of the Harder-Narasimhan filtration. Therefore, the short exact sequence

$$0 \to \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), E/N^{k-1}) \to \mathrm{Hom}(N^{j+1}, E/N^{k-1}) \to \mathrm{Hom}(N^j, E/N^{k-1}) \to 0$$

induces an injection

$$H^1(C, \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), E/N^{k-1})) \to H^1(C, \mathrm{Hom}(N^{j+1}, E/N^{k-1})).$$

Hence $T_C(-D) \to \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), E/N^{k-1})$ induces the zero map on H^1, since we have seen above the composite $T_C(-D) \to \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), E/N^{k-1}) \to \mathrm{Hom}(N^{j+1}, E/N^{k-1})$ vanishes on H^1.

We conclude by showing the map

$$\phi_{j+1,k} : T_C(-D) \to \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), \mathrm{gr}_N^k(E))$$

vanishes on H^1. Since the slopes of the Harder-Narasimhan constituents of $\mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), E/N^k)$ are negative, we find

$$H^0(C, \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), E/N^k)) = 0.$$

The short exact sequence

$$0 \to \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), \mathrm{gr}_N^k(E)) \to \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), E/N^{k-1}) \to \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), E/N^k) \to 0$$

therefore induces an injection

$$H^1(C, \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), \mathrm{gr}_N^k(E))) \to H^1(C, \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), E/N^{k-1})).$$

Therefore, the map $\phi_{j+1,k} : T_C(-D) \to \mathrm{Hom}(\mathrm{gr}_N^{j+1}(E), \mathrm{gr}_N^k(E)) = E_{j+1,k}$ induces zero on H^1 as desired. □

We now show that if the map $H^1(C, T_C(-D)) \to H^1(C, E_{j+1,k})$ vanishes, we will be able to produce a vector bundle which is not generically globally generated. We will later apply Proposition 5.1.3 and Lemma 5.1.2 to this vector bundle to obtain Theorem 1.3.4(1) and (2).

Lemma 5.2.6. *With notation as in Notation 5.2.1, suppose the map $H^1(C, T_C(-D)) \to H^1(C, E_{j+1,k})$ (induced by the non-zero map $\phi_{j+1,k} : T_C(-D) \to E_{j+1,k}$ of Lemma 5.2.3) vanishes. Then the $E^\vee_{j+1,k} \otimes \omega_C$ is not generically globally generated.*

Proof. Since $\phi_{j+1,k} : T_C(-D) \to E_{j+1,k}$ is nonzero, we obtain a nonzero Serre dual map

$$E^\vee_{j+1,k} \otimes \omega_C \to \omega_C^{\otimes 2}(D), \tag{5.3}$$

which induces the 0 map

$$H^0(C, E^\vee_{j+1,k} \otimes \omega_C) \to H^0(C, \omega_C^{\otimes 2}(D)).$$

In particular, $E^\vee_{j+1,k} \otimes \omega_C$ is not generically globally generated. Indeed, any global section must lie in the kernel of (5.3), which has corank one in $E^\vee_{j+1,k} \otimes \omega_C$. □

We now prove Theorem 1.3.4.

5.2.7.

Proof of Theorem 1.3.4. We use notation as in Notation 5.2.1. We aim first to show that if (E', ∇') is the isomonodromic deformation of (E, ∇) to an analytically general nearby curve C', then for every i there are some $j < i < k$ with $\mathrm{rk}\, \mathrm{gr}_{HN}^{j+1} E' \cdot \mathrm{rk}\, \mathrm{gr}_{HN}^k E' \geq g + 1$.

By [BHH21, Lemma 5.1], the locus of bundles in a family $\mathscr{E}$ on $\mathscr{C} \to \Delta$ which are not semistable form a closed analytic subset, and if a general member is not semistable, then, after passing to an open subset of Δ, there is a filtration on $\mathscr{E}$ restricting to the Harder-Narasimhan filtration on each fiber. Thus after replacing (C, D) with an analytically general nearby curve (C', D'), and replacing (E, ∇) with the restriction (E', ∇') of the isomonodromic deformation to (C', D'), we may assume the Harder-Narasimhan filtration $HN^\bullet$ of E' extends to a filtration of $\mathscr{E}$ on a first-order neighborhood of C'.

We next verify that for every $0 < i < n$, there is some $j < i < k$ for which $E'^\vee_{j+1,k} \otimes \omega_{C'}$ is not generically globally generated. By Lemma 5.2.3, for every $0 < i < n$, there is some $j < i$ and $k \geq i+1$ so that the map $T_{C'}(-D') \to \operatorname{End}(E') / \operatorname{End}_{HN^\bullet}(E')$ induces a nonzero map

$$T_{C'}(-D') \to E'_{j+1,k} := \operatorname{Hom}(\operatorname{gr}^{j+1}_{HN} E', \operatorname{gr}^k_{HN} E').$$

By Lemma 5.2.5, $H^1(C', T_{C'}(-D')) \to H^1(C', E'_{j+1,k})$ vanishes. By Lemma 5.2.6, $E'^{\vee}_{j+1,k} \otimes \omega_{C'}$ is not generically globally generated.

We are finally in a position to prove Theorem 1.3.4(1). It follows from Proposition 5.1.3 that

$$\operatorname{rk} \operatorname{gr}^{j+1}_{HN} E' \cdot \operatorname{rk} \operatorname{gr}^k_{HN} E' = \operatorname{rk} E'^{\vee}_{j+1,k} \otimes \omega_{C'} \geq g+1.$$

Thus Theorem 1.3.4(1) holds.

We now conclude by verifying Theorem 1.3.4(2). By Lemma 5.1.2, we have that

$$E'^{\vee}_{j+1,k} \otimes \omega_{C'} = \operatorname{Hom}(\operatorname{gr}^{j+1}_{HN} E', \operatorname{gr}^k_{HN} E')^{\vee} \otimes \omega_{C'}$$

must have slope at most $2g-1$ since it is not generically globally generated. As $\operatorname{Hom}(\operatorname{gr}^{j+1}_{HN} E', \operatorname{gr}^k_{HN} E')$ has negative slope by the definition of the Harder-Narasimhan filtration,

$$-1 \leq \mu(\operatorname{Hom}(\operatorname{gr}^{j+1}_{HN}(E'), \operatorname{gr}^k_{HN}(E'))) < 0.$$

Since the slope of a tensor product of vector bundles is the sum of their slopes, we find

$$0 < \mu(\operatorname{gr}^{j+1}_{HN}(E')) - \mu(\operatorname{gr}^k_{HN}(E')) \leq 1.$$

Using $\mu(\operatorname{gr}^{j+1}_{HN}(E')) \geq \mu(\operatorname{gr}^i_{HN}(E'))$ as $j+1 \leq i$ and $\mu(\operatorname{gr}^{i+1}_{HN}(E')) \geq \mu(\operatorname{gr}^k_{HN}(E'))$ as $i+1 \leq k$, we conclude $0 < \mu(\operatorname{gr}^i_{HN}(E')) - \mu(\operatorname{gr}^{i+1}_{HN}(E')) \leq 1$. □

5.2.8. We next prove Corollary 1.3.6, which follows from Theorem 1.3.4 and the AM-GM inequality.

Proof of Corollary 1.3.6. If (E', Δ') is an isomonodromic deformation of (E, Δ) to an analytically general nearby curve which is not semistable, it follows from Theorem 1.3.4(1) that there will be j, k with $j < i < k$ so that the Harder-Narasimhan filtration HN of E' satisfies $\operatorname{rk} \operatorname{gr}^{j+1}_{HN} E' \cdot \operatorname{rk} \operatorname{gr}^k_{HN} E' \geq g+1$. Since $\operatorname{rk} \operatorname{gr}^{j+1}_{HN} E' + \operatorname{rk} \operatorname{gr}^k_{HN} E' \leq \operatorname{rk} E' = \operatorname{rk} E$, it follows from the arithmetic mean-geometric mean inequality that

$$g+1 \leq \operatorname{rk} \operatorname{gr}^{j+1}_{HN} E' \cdot \operatorname{rk} \operatorname{gr}^k_{HN} E' \leq \left(\frac{\operatorname{rk} E}{2}\right)^2.$$

So $\operatorname{rk} E \geq 2\sqrt{g+1}$ as desired. □

6. Variations of Hodge structure on an analytically general curve

In this section we prove Theorem 1.2.8, Theorem 1.2.4, and Corollary 1.2.6.

6.1. The proof of Theorem 1.2.8.

We start with the following lemma, which gives a useful criterion for showing a representation is unitary.

Lemma 6.1.1. *Suppose $(C, x_1, \ldots, x_n)$ is an n-pointed hyperbolic curve and (E, ∇) is a flat vector bundle on $C \setminus \{x_1, \ldots, x_n\}$ underlying a polarizable complex variation of Hodge structure with unipotent monodromy around the x_i. Let $(\overline{E}, \overline{\nabla})$ be the Deligne canonical extension of (E, ∇) to C. If $\overline{E}$ is semistable, then the representation of $\pi_1(C \setminus \{x_1, \cdots, x_n\})$ associated to (E, ∇) is unitary.*

Proof. By Proposition 3.1.5(2), we may write $\mathbb{V} := \ker(\nabla)$ as

$$\mathbb{V} := \bigoplus_i \mathbb{L}_i \otimes W_i$$

where the $\mathbb{L}_i$ each have irreducible monodromy, unipotent monodromy about the x_i, and also underlie polarizable variations of Hodge structure, and the W_i are constant complex Hodge structures. It suffices to show the representation associated to each $\mathbb{L}_i$ has unitary monodromy. We may therefore reduce to the case that $\mathbb{V} = \mathbb{L}_i$ and assume that (E, ∇) has irreducible monodromy.

Let i be maximal such that $F^i\overline{E}$ is non-zero. Since $\overline{E}$ is semistable, it follows from Corollary 3.1.9 that the natural map

$$F^i\overline{E} \to F^{i-1}\overline{E}/F^i\overline{E} \otimes \omega_C$$

induced by the connection is zero, i.e. the connection preserves $F^i\overline{E}$. By irreducibility of the monodromy of (E, ∇), we must have that $F^i\overline{E}$ equals $\overline{E}$. But in this case (E, ∇) is unitary, as the monodromy preserves the polarization, a definite Hermitian form. □

6.1.2. We now recall the setup of Theorem 1.2.8. Let $(C, x_1, \cdots, x_n)$ be an n-pointed hyperbolic curve of genus g. Let (E, ∇) be a flat vector bundle on C with $\operatorname{rk} E < 2\sqrt{g+1}$ such that (E, ∇) has regular singularities and nilpotent residues at the x_i (i.e. it is the Deligne canonical extension to C of its restriction to $C \setminus \{x_1, \cdots, x_n\}$). Our goal is to show that if an isomonodromic deformation of (E, ∇) to an analytically general nearby n-pointed curve underlies a polarizable complex variation of Hodge structure, then (E, ∇) has unitary monodromy.

Proof of Theorem 1.2.8. After replacing $(C, x_1, \cdots, x_n)$ with an analytically general nearby curve, and (E, ∇) with an isomonodromic deformation to this curve, we may assume by Corollary 1.3.6 that E is semistable. Thus (E, ∇) has unitary monodromy by Lemma 6.1.1. ☐

6.1.3. *The proof of Theorem 1.2.4.* The proof of Theorem 1.2.4 follows from the integrality assumption and the following lemma.

Lemma 6.1.4. *Suppose Γ is a group, K is a number field, and ρ is a representation*

$$\rho : \Gamma \to \mathrm{GL}_m(\mathscr{O}_K).$$

If for each embedding $\iota : K \hookrightarrow \mathbb{C}$ the representation $\rho \otimes_{\mathscr{O}_K, \iota} \mathbb{C}$ is unitary, then ρ has finite image.

Proof. Indeed, for $\iota : K \hookrightarrow \mathbb{C}$ an embedding, let

$$\rho_\iota : \Gamma \to \mathrm{GL}_m(\mathbb{C})$$

be the corresponding representation $\rho \otimes_{\mathscr{O}_K, \iota} \mathbb{C}$. First,

$$\prod_\iota \rho_\iota : \Gamma \to \prod_\iota \mathrm{GL}_m(\mathbb{C})$$

has compact image by the definition of unitarity. Moreover, the image of

$$\mathscr{O}_K \hookrightarrow \prod_\iota \mathbb{C}$$

is discrete, since the difference of any two distinct elements of $\mathscr{O}_K$ has norm at least 1. Hence the image of $\prod_\iota \rho_\iota$ is discrete and compact, and therefore finite. ☐

Let K be a number field with ring of integers $\mathscr{O}_K$. Let $(C, x_1, \cdots, x_n)$ be an analytically general hyperbolic n-pointed curve of genus g, and let $\mathbb{V}$ be a $\mathscr{O}_K$-local system on $C \setminus \{x_1, \cdots, x_n\}$ with infinite monodromy, and unipotent monodromy about the x_i. Suppose that for each embedding $\iota : \mathscr{O}_K \hookrightarrow \mathbb{C}$, $\mathbb{V} \otimes_{\mathscr{O}_K, \iota} \mathbb{C}$ underlies a polarizable complex variation of Hodge structure. Our goal is to prove Theorem 1.2.4, which states that

$$\mathrm{rk}_{\mathscr{O}_K}(\mathbb{V}) \geq 2\sqrt{g+1}.$$

Proof of Theorem 1.2.4. We use $\mathscr{T}_{g,n}$ to denote the universal cover of $\mathscr{M}_{g,n}$. For a fixed representation $\rho : \pi_1(C \setminus \{x_1, \cdots, x_n\})) \to \mathrm{GL}_m(\mathscr{O}_K)$ let T_ρ denote the set of $[(C', x'_1, \cdots, x'_n)] \in \mathscr{T}_{g,n}$, for which the associated $\mathscr{O}_K$-local system $\mathbb{V}$ has the following property: for each embedding $\iota : \mathscr{O}_K \hookrightarrow \mathbb{C}$, $\mathbb{V} \otimes_{\mathscr{O}_K, \iota} \mathbb{C}$ underlies a polarizable complex variation of Hodge structure on $C' \setminus \{x'_1, \ldots, x'_n\}$. Let M_ρ denote the image of T_ρ under the covering map $\mathscr{T}_{g,n} \to \mathscr{M}_{g,n}$.

Our goal is to show that an analytically very general point of $\mathscr{M}_{g,n}$ lies in the complement of the union of the M_ρ, where ρ ranges over the set of representations of $\pi_1(C \setminus \{x_1, \cdots, x_n\}) \to \mathrm{GL}_r(\mathscr{O}_K)$, with infinite image, for

K a number field, and $r < 2\sqrt{g+1}$. Since there are only countably many such representations ρ, it is enough to show that an analytically very general point lies in the complement of M_ρ. Since $(C, x_1, \ldots, x_n)$ is hyperbolic, $\mathscr{T}_{g,n} \to \mathscr{M}_{g,n}$ is a covering space of countable degree, and so the image of a closed analytic set is locally contained in a countable union of closed analytic subsets. It therefore suffices to show that for any ρ with infinite monodromy and rank $< 2\sqrt{g+1}$, T_ρ is contained in a closed analytic subset of $\mathscr{T}_{g,n}$.

We now show such T_ρ as above are contained in a closed analytic subset of $\mathscr{T}_{g,n}$. Indeed, suppose $\mathbb{V}$ is the local system associated to ρ on some curve $C \setminus \{x_1, \cdots, x_n\}$, and that for each embedding $\iota : \mathscr{O}_K \hookrightarrow \mathbb{C}$, $\mathbb{V} \otimes_{\mathscr{O}_K, \iota} \mathbb{C}$, underlies a polarizable complex variation of Hodge structure. It is enough to show this complex polarizable variation of Hodge structure does not extend to an analytically general nearby curve. Indeed, if it did, Theorem 1.2.8 implies $\prod_{\iota : \mathscr{O}_K \to \mathbb{C}} \rho_\iota$ has unitary monodromy, and Lemma 6.1.4 implies its monodromy is finite. This contradicts our assumption that ρ has infinite monodromy. □

6.1.5.

Proof of Corollary 1.2.6. Let $g \geq 2$ be an integer and let $(C, x_1, \cdots, x_n)$ be an analytically general hyperbolic n-pointed curve of genus g. Let $f : X \to C \setminus \{x_1, \cdots, x_n\}$ be a smooth proper morphism, $i \geq 0$ is an integer, and $\mathbb{V} \subset R^i f_* \mathbb{C}$ is a sub-local system with infinite monodromy and unipotent monodromy about the x_i. Then we wish to show that $\dim_{\mathbb{C}} \mathbb{V} \geq 2\sqrt{g+1}$.

It suffices to show that $\mathbb{V}$ satisfies the hypotheses of Theorem 1.2.4. The existence of an $\mathscr{O}_K$-structure follows from the fact that $R^i f_* \mathbb{C}$ has a $\mathbb{Z}$-structure. Let $\mathbb{W}$ be the corresponding $\mathscr{O}_K$-local system. All that remains is to verify that for each embedding $\iota : \mathscr{O}_K \hookrightarrow \mathbb{C}$, the corresponding complex local system $\mathbb{W} \otimes_{\mathscr{O}_K, \iota} \mathbb{C}$ underlies a polarizable complex variation of Hodge structure. But each such embedding yields a summand of $R^i f_* \mathbb{C}$, Galois-conjugate to the original embedding $\mathbb{W} \subset \mathbb{V} \subset R^i f_* \mathbb{C}$. Any such summand underlies a polarizable variation, by Proposition 3.1.5(2), which completes the proof. □

7. Questions

7.1. **Bounds.**

Question 7.1.1. Is the bound of $2\sqrt{g+1}$ appearing in Corollary 1.3.6, Theorem 1.2.4 and Theorem 1.2.8 sharp? If not, can one explicitly construct low rank geometric variations of Hodge structure with infinite monodromy on a general curve or general n-pointed curve? Do there exist counterexamples to the above results if one replaces $2\sqrt{g+1}$ by a linear function of g?

We have no reason to believe the bound is sharp. The Kodaira-Parshin trick (as used in §4, for example) is one source of variations of Hodge structure on $\mathscr{M}_{g,n}$ of rank bounded in terms of g, n, but it is not the only one. For example, the representations constructed in [KS16] are cohomologically rigid and hence underlie integral variations of Hodge structure by [EG18, Theorem 1.1] and [Sim92, Theorem 3]; on Simpson's motivicity conjecture ([Sim92, Conjecture, p. 9]) they are geometric in nature, though this is not clear from the construction. Of course it would be extremely interesting to prove that these representations arise from algebraic geometry.

It appears that the representations constructed in [KS16] have rank growing exponentially in g; it is natural, given our results, to ask if one can use their methods to produce representations of smaller rank.

We also raise a related question about bounds on maps to the moduli space of curves.

Question 7.1.2. Fix an integer $g \geq 2$. What is the smallest integer $h \geq 2$ for which the generic genus g curve, i.e., the generic fiber of $\mathscr{M}_{g,1} \to \mathscr{M}_g$, has a non-constant map to $\mathscr{M}_h$?

Remark 7.1.3. Since a map $C \to \mathscr{M}_h$ corresponds to a family of semistable curves of genus h over C, by considering the associated family of Jacobians, it follows from Corollary 1.2.7 that $h \geq \sqrt{g+1}$. The Kodaira-Parshin trick Proposition 4.1.1 does not a priori apply to construct maps from the generic curve to $\mathscr{M}_h$, because as written it produces disconnected covers. But one can apply a variant where one takes a cover defined by a characteristic quotient of the fundamental group to show there is some (fairly large) value of h for which the generic genus g curve has a non-constant map to $\mathscr{M}_h$. See [McM00, Theorem 1.4] for more details.

7.2. **A parabolic variant.** There are variants for all of the background appearing in §2 and §3 when $(V, V^{p,q}, D)$ in Proposition 3.1.5 is only assumed to have *quasi-unipotent* monodromy about the components of Z, where we replace Chern classes with *parabolic Chern classes*, polystability with *parabolic polystability*, and so on.

Question 7.2.1. Can one extend the results of this paper to the parabolic setting?

Remark 7.2.2. We suspect such an extension would likely come at the cost of introducing a dependency of the bounds appearing in the main theorems on the number of parabolic points. It may also be cleaner to phrase some of the extensions in the equivalent language of stacky curves.

7.3. **Non-abelian Hodge loci.** Let $(C, x_1, \cdots, x_n)$ be an n-pointed curve and $\mathbb{V}$ a $\mathbb{Z}$-local system on $C \setminus \{x_1, \cdots, x_n\}$ with (quasi)-unipotent monodromy about the x_i, and let $(\mathscr{E}, \nabla)$ be the associated flat vector bundle. We refer to the locus $H_{\mathbb{V}}$ in $\mathscr{T}_{g,n}$ where the corresponding isomonodromic deformation of $(\mathscr{E}, \nabla)$ underlies a polarizable variation of Hodge structure as an *non-abelian Hodge locus*. By analogy to the famous result on algebraicity of Hodge loci of Cattani-Deligne-Kaplan [CDK95], it is natural to ask:

Question 7.3.1 (Compare to [Sim97, Conjecture 12.3])**.** Let Z be an irreducible component of $H_{\mathbb{V}}$. Is the image of Z in $\mathscr{M}_{g,n}$ algebraic?

This would follow if all $\mathbb{Z}$-local systems which underlie polarizable variations of Hodge structure arise from geometry, which is perhaps a folk conjecture (and is conjectured explicitly in [Sim97, Conjecture 12.4]). Just as [CDK95] provides evidence for the Hodge conjecture, a positive answer to Question 7.3.1 would provide evidence for this conjecture.

When we refer to an analytically very general curve, in Theorem 1.2.4, we mean in the sense of Definition 1.2.2. A positive answer to Question 7.3.1 would allow us to replace this with the usual algebraic notion of a very general curve in Theorem 1.2.4. It seems plausible that one can make this replacement in Corollary 1.2.6 without requiring input from Question 7.3.1, using the main result of [CDK95].

REFERENCES

[Ara19] Donu Arapura. Kodaira–Saito vanishing via Higgs bundles in positive characteristic. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2019(755):293–312, 2019.

[Ati57] Michael Francis Atiyah. Complex analytic connections in fibre bundles. *Transactions of the American Mathematical Society*, 85(1):181–207, 1957.

[BHH16] Indranil Biswas, Viktoria Heu, and Jacques Hurtubise. Isomonodromic deformations of logarithmic connections and stability. *Math. Ann.*, 366(1-2):121–140, 2016.

[BHH17] Indranil Biswas, Viktoria Heu, and Jacques Hurtubise. Isomonodromic deformations and very stable vector bundles of rank two. *Comm. Math. Phys.*, 356(2):627–640, 2017.

[BHH20] Indranil Biswas, Viktoria Heu, and Jacques Hurtubise. Isomonodromic deformations of logarithmic connections and stable parabolic vector bundles. *Pure Appl. Math. Q.*, 16(2):191–227, 2020.

[BHH21] Indranil Biswas, Viktoria Heu, and Jacques Hurtubise. Isomonodromic deformations of irregular connections and stability of bundles. *Comm. Anal. Geom.*, 29(1):1–18, 2021.

[Bol95] AA Bolibruch. The Riemann-Hilbert problem and Fuchsian differential equations on the Riemann sphere. In *Proceedings of the International Congress of Mathematicians*, pages 1159–1168. Springer, 1995.

[BPGN97] L. Brambila-Paz, I. Grzegorczyk, and P. E. Newstead. Geography of Brill-Noether loci for small slopes. *J. Algebraic Geom.*, 6(4):645–669, 1997.

[CD17] Fabrizio Catanese and Michael Dettweiler. Answer to a question by Fujita on variation of Hodge structures. In *Higher dimensional algebraic geometry—in honour* *of Professor Yujiro Kawamata's sixtieth birthday*, volume 74 of *Adv. Stud. Pure Math.*, pages 73–102. Math. Soc. Japan, Tokyo, 2017.

[CDK95] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan. On the locus of Hodge classes. *Journal of the American Mathematical Society*, 8(2):483–506, 1995.

[CKS86] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid. Degeneration of Hodge structures. *Annals of Mathematics*, 123(3):457–535, 1986.

[CMSP17] James Carlson, Stefan Müller-Stach, and Chris Peters. *Period mappings and period domains*, volume 168 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. Second edition of [MR2012297].

[Del70] Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[Del71] Pierre Deligne. Théorie de Hodge: II. *Publications Mathématiques de l'IHÉS*, 40:5–57, 1971.

[Del87] Pierre Deligne. Un théoreme de finitude pour la monodromie. In *Discrete groups in geometry and analysis*, pages 1–19. Springer, 1987.

[dF13] Michele de Franchis. Un teorema sulle involuzioni irrazionali. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 36(1):368–368, 1913.

[Dor01] Charles F. Doran. Algebraic and geometric isomonodromic deformations. *J. Differential Geom.*, 59(1):33–85, 2001.

[EG18] Hélène Esnault and Michael Groechenig. Cohomologically rigid local systems and integrality. *Selecta Mathematica*, 24(5):4279–4292, 2018.

[EV86] Hélene Esnault and Eckart Viehweg. Logarithmic de Rham complexes and vanishing theorems. *Inventiones mathematicae*, 86(1):161–194, 1986.

[EV99] Hélene Esnault and Eckart Viehweg. Semistable bundles on curves and irreducible representations of the fundamental group. *Contemporary mathematics*, 241:129–140, 1999.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[Gri70] Phillip A Griffiths. Periods of integrals on algebraic manifolds, III (Some global differential-geometric properties of the period mapping). *Publications Mathématiques de l'IHÉS*, 38:125–180, 1970.

[Gro65] Alexandre Grothendieck. Éléments de géométrie algébrique IV. *Publ. Math. IHES, Ibid.*, 24, 1965.

[Gro72] A. Grothendieck. *Groupes de monodromie en géométrie algébrique. Tome 1*. Séminaire de Géométrie Algébrique du Bois Marie 1967–1969 (SGA 7). Lecture Notes in Mathematics, Vol. 288. Springer-Verlag, Berlin-New York, 1972.

[Heu10] Viktoria Heu. Universal isomonodromic deformations of meromorphic rank 2 connections on curves. *Ann. Inst. Fourier (Grenoble)*, 60(2):515–549, 2010.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[KS16] Thomas Koberda and Ramanujan Santharoubane. Quotients of surface groups and homology of finite covers via quantum representations. *Inventiones mathematicae*, 206(2):269–292, 2016.

[LV20] Brian Lawrence and Akshay Venkatesh. Diophantine problems and p-adic period mappings. *Invent. Math.*, 221(3):893–999, 2020.

[Mal83a] B. Malgrange. Sur les déformations isomonodromiques. I. Singularités régulières. In *Mathematics and physics (Paris, 1979/1982)*, volume 37 of *Progr. Math.*, pages 401–426. Birkhäuser Boston, Boston, MA, 1983.

[Mal83b] B. Malgrange. Sur les déformations isomonodromiques. II. Singularités irrégulières. In *Mathematics and physics (Paris, 1979/1982)*, volume 37 of *Progr. Math.*, pages 427–438. Birkhäuser Boston, Boston, MA, 1983.

[McM00] Curtis T. McMullen. From dynamics on surfaces to rational points on curves. *Bull. Amer. Math. Soc. (N.S.)*, 37(2):119–140, 2000.

[Par68] A. N. Paršin. Algebraic curves over function fields. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 32:1191–1219, 1968.

[Pet00] Chris Peters. Arakelov-type inequalities for hodge bundles. *arXiv preprint math/0007102*, 2000.

[SA66] Pierre Samuel and Sivaramakrishna Anantharaman. *Lectures on old and new results on algebraic curves*, volume 36. Tata Institute of Fundamental Research Bombay, 1966.

[Ser06] Edoardo Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[Sim88] Carlos T Simpson. Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. *Journal of the American Mathematical Society*, pages 867–918, 1988.

[Sim90] Carlos T Simpson. Harmonic bundles on noncompact curves. *Journal of the American Mathematical Society*, 3(3):713–770, 1990.

[Sim92] Carlos T Simpson. Higgs bundles and local systems. *Publications Mathématiques de l'IHÉS*, 75:5–95, 1992.

[Sim97] Carlos T Simpson. The Hodge filtration on nonabelian cohomology. *Algebraic geometry-Santa Cruz 1995, Proc. Sympos. Pure Math., Part 2*, 62:217–281, 1997.