Tate's Thesis: L-Functions and Harmonic Analysis on the Adeles

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Motivation: Riemann Zeta Function

Consider the Riemann zeta function

$$\zeta(s) = \sum_{\substack{n \subseteq \mathbb{Z} \\ \text{ideal}}} \frac{1}{n^s} = \prod_{\substack{p \subseteq \mathbb{Z} \\ \text{prime}}} \frac{1}{1 - p^{-s}} \qquad \Re(s) > 1$$

Let $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$.

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Let $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$.

Theorem

The completed zeta function $Z(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$ admits an **analytic continuation** with a simple pole at $s = 1$ with residue 1, and satisfies the **functional equation**

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\tfrac{1-s}{2})\zeta(1-s)$$

Motivation: Dedekind Zeta Function

Let K be a number field with r_1 and r_2 real and complex places and discriminant Δ_K. Consider the Dedekind zeta function

$$\zeta_K(s) = \sum_{\substack{\mathfrak{n} \subseteq \mathcal{O}_K \\ \text{ideal}}} \frac{1}{\mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{n})^s} = \prod_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ \text{prime}}} \frac{1}{1 - \mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{p})^{-s}} \qquad \Re(s) > 1$$

Let $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

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Theorem

The completed zeta function $Z_K(s) = |\Delta_K|^{-\frac{s}{2}}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s)$ admits an **analytic continuation** with a simple pole at $s = 1$ with residue κ_K given by the class number formula, and satisfies the **functional equation**

$$|\Delta_K|^{-\frac{s}{2}}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s) = |\Delta_K|^{-\frac{1-s}{2}}\Gamma_{\mathbb{R}}(1-s)^{r_1}\Gamma_{\mathbb{C}}(1-s)^{r_2}\zeta_K(1-s)$$

Motivation: Dirichlet L-Functions

Let $\chi : \mathbb{Z} \to \mathbb{C}$ be a Dirichlet character with modulus N. Consider the Dirichlet L-function

$$\ell(\chi, s) = \sum_{\substack{n \subseteq \mathbb{Z} \\ \text{ideal}}} \frac{\chi(n)}{n^s} = \prod_{\substack{p \subseteq \mathbb{Z} \\ \text{prime}}} \frac{1}{1 - \chi(p) p^{-s}} \qquad \Re(s) > 1$$

Let $\Gamma_{\mathbb{R}}(\chi, s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$ for $\chi(-1) = 1$ (even) and $\Gamma_{\mathbb{R}}(\chi, s) = \pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2})$ for $\chi(-1) = -1$ (odd).

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Theorem

The completed L-function $L(\chi, s) = N^{\frac{s}{2}} \Gamma_{\mathbb{R}}(\chi, s) \ell(\chi, s)$ admits an **analytic continuation** with a simple pole at $s = 1$ with residue $\frac{\varphi(N)}{N}$ when χ is principal, and satisfies the **functional equation**

$$N^{\frac{s}{2}} \Gamma_{\mathbb{R}}(\chi, s) \ell(s, \chi) = \varepsilon(\chi) N^{\frac{1-s}{2}} \Gamma_{\mathbb{R}}(\chi, 1 - s) \ell(\chi^{-1}, 1 - s)$$

with $|\varepsilon(\chi)| = 1$, $\varepsilon(\chi)\varepsilon(\chi^{-1}) = 1$.

Adeles

Let K be a number field with ring of integers $\mathcal{O}_K$. For v a place of K let K_v be the completion of K at v with valuation ring $\mathcal{O}_v$. Let $\mathbb{A}_K$ be the adele ring of K:

$$\mathbb{A}_K = \widehat{\prod_{v \nmid \infty}}^{\mathcal{O}_v} K_v \times \prod_{v \mid \infty} K_v$$

Consisting of $(x_v)_v \in \prod_v K_v$ such that $x_v \in \mathcal{O}_v$ for all but finitely many finite places v of K.

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We have **diagonal embeddings** $K \subseteq \mathbb{A}_K$ and $K^\times \subseteq \mathbb{I}_K$.

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Adeles

The local fields K_v with the usual topology are locally compact Abelian groups which are **Pontryagin self-dual** in the sense that $\widehat{K}_v = \mathrm{Hom}(K_v, \mathrm{U}(1)) = K_v$. Consider the restricted product topology on the adele ring $\mathbb{A}_K$ generated by restricted open rectangles

$$U = \prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_v$$

for S a finite subset of the set of places of K, with $U_v \subseteq K_v$ open. Then $\mathbb{A}_K$ is a locally compact Abelian group which is **Pontryagin self-dual** in the sense that $\widehat{\mathbb{A}}_K = \mathrm{Hom}(\mathbb{A}_K, \mathrm{U}(1)) = \mathbb{A}_K$.

Adeles

Let $\mathbb{I}_K^+ = \{x \in \mathbb{I}_K \mid |x| \geq 1\}$, $\mathbb{I}_K^- = \{x \in \mathbb{I}_K \mid |x| \leq 1\}$ and $\mathbb{I}_K^1 = \mathbb{I}_K^+ \cap \mathbb{I}_K^- = \{x \in \mathbb{I}_K \mid |x| = 1\}$.
The quotient $\mathbb{I}_K^1/K^\times$ is **compact** (Fujisaki's compactness lemma), which implies finiteness of class number for K and Dirichlet's unit theorem for $\mathcal{O}_K$.

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Theorem

Let $\mathrm{d}^\times x$ be a Haar measure on $\mathbb{I}_K$ normalized with $\mu(\mathbb{I}_K/K^\times) = 1$. Then the volume $\mu(\mathbb{I}_K^1/K^\times)$ is given by the class number formula

$$\mu(\mathbb{I}_K^1/K^\times) = \kappa_K = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{\sqrt{|\Delta_K|}\, w_K}$$

where h_K is the class number of K, where R_K is the Borel regulator of K, where w_K is the number of roots of unity in K.

Schwartz Functions

Definition

For v an Archimedean place of K let $\mathcal{S}(K_v)$ denote the space of **Schwartz functions** on K_v: the $\mathbb{C}$-vector space of smooth functions $f : K_v \to \mathbb{C}$ such that $f^{(n)} : K_v \to \mathbb{C}$ has rapid decay for all $n \geq 0$.

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Definition

Let $\mathcal{S}(\mathbb{A}_K)$ denote the space of **Bruhat-Schwarz functions** on $\mathbb{A}_K$: the $\mathbb{C}$-vector space of finite $\mathbb{C}$-linear combinations of **monomial Bruhat-Schwartz functions** $f : \mathbb{A}_K \to \mathbb{C}$, namely $f = \prod_v f_v$ for $f_v \in \mathcal{S}(K_v)$ with $f_v = \chi_{\mathcal{O}_v}$ for all but finitely many finite places v of K.

Global Fourier Transform

Let $\chi : \mathbb{A}_K \to \mathbb{C}^\times$ be a nontrivial character trivial on the diagonal $K \subseteq \mathbb{A}_K$, yielding a nontrivial character $\chi : \mathbb{A}_K/K \to \mathbb{C}^\times$.

Definition

For $f \in \mathcal{S}(\mathbb{A}_K)$ define the **global Fourier transform**

$$\widehat{f}(\xi) = \int_{\mathbb{A}_K} \overline{\chi}(\xi x) f(x) \mathrm{d}x \qquad \left(f(x) = \int_{\mathbb{A}_K} \chi(\xi x) \widehat{f}(\xi) \mathrm{d}\xi \right)$$

with Haar measure $\mathrm{d}x$ and $\mathrm{d}\xi$ normalized with $\mu(\mathbb{A}_K/K) = 1$.

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with Haar measure $\mathrm{d}x$ and $\mathrm{d}\xi$ normalized with $\mu(\mathbb{A}_K/K) = 1$.

Theorem

(Global Poisson Summation) For $f \in \mathcal{S}(\mathbb{A}_K)$ we have

$$\boxed{\sum_{x \in K} f(x) = \sum_{\xi \in K} \widehat{f}(\xi)}$$

Global Theta Functions

For $f \in \mathcal{S}(\mathbb{A}_K)$ consider the **global theta function**

$$\theta_f(x) = \sum_{\alpha \in K} f(\alpha x)$$

With change of measure $\mathrm{d}_v^\times(\frac{\alpha}{x}) = \frac{1}{|x|_v}\mathrm{d}_v^\times \alpha$ hence $\mathrm{d}^\times(\frac{\alpha}{x}) = \frac{1}{|x|}\mathrm{d}^\times \alpha$

$$\int_{\mathbb{A}_K} \overline{\chi}(\xi\alpha) f(\alpha x)\mathrm{d}\alpha = \frac{1}{|x|}\int_{\mathbb{A}_K} \overline{\chi}(\xi \tfrac{\alpha}{x}) f(\alpha)\mathrm{d}\alpha = \frac{1}{|x|}\widehat{f}(\tfrac{\xi}{x})$$

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We obtain the **functional equation** for the global theta function

$$\boxed{\theta_f(x) = \sum_{\alpha \in K} f(\alpha x) = \frac{1}{|x|}\sum_{\alpha \in K} \widehat{f}(\frac{\alpha}{x}) = \frac{1}{|x|}\theta_{\widehat{f}}(\frac{1}{x})}$$

Global Zeta Integrals

Let $\chi : \mathbb{I}_K \to \mathbb{C}^\times$ be a character trivial on the diagonal $K^\times \subseteq \mathbb{I}_K$, yielding a Hecke character $\chi : \mathbb{I}_K / K^\times \to \mathbb{C}^\times$. Suppose moreover that χ is trivial on the diagonal $\mathbb{R}_{>0}$ in Archimedean places in $\mathbb{I}_K$.

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Definition

For $f \in \mathcal{S}(\mathbb{A}_K)$ define the global zeta integral

$$Z_f(\chi, s) = \int_{\mathbb{I}_K} |x|^s \chi(x) f(x) \mathrm{d}^\times x \qquad \Re(s) > 1$$

with Haar measure $\mathrm{d}^\times x$ normalized with $\mu(\mathbb{I}_K / K^\times) = 1$.

Global Zeta Integrals

Definition

For v a place of K and $f_v \in \mathcal{S}(K_v)$ define the local zeta integral

$$Z_{f_v}(\chi_v, s) = \int_{K_v^\times} |x|_v^s \chi_v(x) f_v(x) \mathrm{d}_v^\times x$$

with Haar measure $\mathrm{d}_v^\times x$ normalized with $\mu(\mathbb{I}_K / K^\times) = 1$.

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with Haar measure $\mathrm{d}_v^\times x$ normalized with $\mu(\mathbb{I}_K / K^\times) = 1$.

For $f = \prod_v f_v$ a monomial Schwartz function

$$\begin{aligned} Z_f(s) &= \int_{\mathbb{I}_K} |x|^s \chi(x) f(x) \mathrm{d}^\times x \\ &= \prod_v \int_{K_v^\times} |x|_v^s \chi_v(x) f_v(x) \mathrm{d}_v^\times x = \prod_v Z_{f_v}(s) \qquad \Re(s) > 1 \end{aligned}$$

Zeta Integrals: Functional Equation

We now analyze the global zeta integral $Z_f(\chi, s)$ in order to establish the **analytic continuation** and **functional equation**. For $x \in \mathbb{I}_K$ and $f \in \mathcal{S}(\mathbb{A}_K)$ let $\theta_f^\times(x) = \theta_f(x) - f(0)$.

$$\begin{aligned}
Z_f(\chi, s) &= \int_{\mathbb{I}_K} |x|^s \chi(x) f(x) \mathrm{d}^\times x = \int_{\mathbb{I}_K/K^\times} \sum_{\alpha \in K^\times} |\alpha x|^s \chi(\alpha x) f(\alpha x) \mathrm{d}^\times(\alpha x) \\
&= \int_{\mathbb{I}_K/K^\times} |x|^s \chi(x) \sum_{\alpha \in K^\times} f(\alpha x) \mathrm{d}^\times x = \int_{\mathbb{I}_K/K^\times} |x|^s \chi(x) \theta_f^\times(x) \mathrm{d}^\times x \\
&= \int_{\mathbb{I}_K^+/K^\times} |x|^s \chi(x) \theta_f^\times(x) \mathrm{d}^\times x + \int_{\mathbb{I}_K^-/K^\times} |x|^s \chi(x) \theta_f^\times(x) \mathrm{d}^\times x
\end{aligned}$$

The integral over $\mathbb{I}_K^+/K^\times$ is entire. For the integral over $\mathbb{I}_K^-/K^\times$ we use the functional equation for the global theta function.

Zeta Integrals: Functional Equation

$$\int_{\mathbb{I}_K^-/K^\times} |x|^s \chi(x) \theta_f^\times(x) \mathrm{d}^\times x = \int_{\mathbb{I}_K^+/K^\times} |\tfrac{1}{x}|^s \chi(\tfrac{1}{x}) \theta_f^\times(\tfrac{1}{x}) \mathrm{d}^\times(\tfrac{1}{x})$$

$$= \int_{\mathbb{I}_K^+/K^\times} |x|^{-s} \chi^{-1}(x) \Big(|x| \theta_{\widehat{f}}(x) - f(0) \Big) \mathrm{d}^\times x = \int_{\mathbb{I}_K^+/K^\times} |x|^{1-s} \chi^{-1}(x) \theta_{\widehat{f}}^\times \mathrm{d}^\times x$$

$$+ \widehat{f}(0) \int_{\mathbb{I}_K^+/K^\times} |x|^s \chi(x) \mathrm{d}^\times x - f(0) \int_{\mathbb{I}_K^+/K^\times} |x|^{1-s} \chi^{-1}(x) \mathrm{d}^\times x$$

The last terms being pole terms.

Zeta Integrals: Functional Equation

$$\int_{\mathbb{I}_K^-/K^\times} |x|^s \chi(x)\theta_f^\times(x)\mathrm{d}^\times x = \int_{\mathbb{I}_K^+/K^\times} |\tfrac{1}{x}|^s \chi(\tfrac{1}{x})\theta_f^\times(\tfrac{1}{x})\mathrm{d}^\times(\tfrac{1}{x})$$

$$= \int_{\mathbb{I}_K^+/K^\times} |x|^{-s}\chi^{-1}(x)\Big(|x|\theta_{\widehat{f}}(x) - f(0)\Big)\mathrm{d}^\times x = \int_{\mathbb{I}_K^+/K^\times} |x|^{1-s}\chi^{-1}(x)\theta_{\widehat{f}}^\times \mathrm{d}^\times x$$

$$+ \widehat{f}(0)\int_{\mathbb{I}_K^+/K^\times} |x|^s\chi(x)\mathrm{d}^\times x - f(0)\int_{\mathbb{I}_K^+/K^\times} |x|^{1-s}\chi^{-1}(x)\mathrm{d}^\times x$$

The last terms being pole terms. The integral over $\mathbb{I}_K^+/K^\times$ is entire. Since $\mu(\mathbb{I}_K^1/K^\times) = 1$ we can evaluate:

$$\widehat{f}(0)\int_{\mathbb{I}_K^+/K^\times} |x|^s\chi(x)\mathrm{d}^\times x = \widehat{f}(0)\mu(\mathbb{I}_K^1/K^\times)\int_1^\infty x^{1-s}\frac{\mathrm{d}x}{x} = \kappa_K\frac{\widehat{f}(0)}{s-1}$$

$$f(0)\int_{\mathbb{I}_K^+/K^\times} |x|^{1-s}\chi^{-1}(x)\mathrm{d}^\times x = f(0)\mu(\mathbb{I}_K^1/K^\times)\int_1^\infty x^{-s}\frac{\mathrm{d}x}{x} = \kappa_K\frac{f(0)}{s}$$

Zeta Integrals: Functional Equation

Putting it all together, we obtain

$$Z_f(\chi, s) = \int_{\mathbb{I}_K^+/K^\times} \left(|x|^s \chi(x) \theta_f^\times(x) + |x|^{1-s} \chi^{-1}(x) \theta_{\widehat{f}}^\times(x) \right) \mathrm{d}^\times x + \kappa_K \frac{\widehat{f}(0)}{s-1} - \kappa_K \frac{f(0)}{s}$$

The integral over $\mathbb{I}_K^+/K^\times$ is entire, hence we have proved **analytic continuation** of $Z_f(\chi, s)$.
The above expression is symmetric in $s \mapsto 1-s$, $f \mapsto \widehat{f}$, and $\chi \mapsto \chi^{-1}$ so we obtain the functional equation

$$Z_f(\chi, s) = Z_{\widehat{f}}(\chi^{-1}, 1-s)$$

hence we have proved the **functional equation** of $Z_f(\chi, s)$.

Riemann Zeta Function: Local Factors

For $v = \infty$ an Archimedean place of $\mathbb{Q}$ take $f_\infty(x) = e^{-\pi x^2}$ the **Gaussian function**. Then $\widehat{f_\infty} = f_\infty$ and we have

$$Z_{f_\infty}(s) = \int_{\mathbb{R}^\times} |x|_\infty^s f_\infty(x) \mathrm{d}_\infty^\times x = \int_{\mathbb{R}} |x|_\infty^{s-1} f_\infty(x) \mathrm{d}_\infty x = \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2})$$

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For $v = p$ a finite place of $\mathbb{Q}$ take $f_p(x) = \chi_{\mathbb{Z}_p}$ the **p-adic Gaussian function**. Then $\widehat{f_p} = f_p$ and we have

$$Z_{f_p}(s) = \int_{\mathbb{Q}_p^\times} |x|_p^s f_p(x) \mathrm{d}_p^\times x = \int_{\mathbb{Z}_p} |x|_p^s \mathrm{d}_p^\times x = \frac{1}{1 - p^{-s}}$$

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For $f = \prod_v f_v$ we obtain the completed Riemann zeta function

$$Z_f(s) = \int_{\mathbb{A}_\mathbb{Q}^\times} |x|^s f(x) \mathrm{d}^\times x = \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \prod_p \frac{1}{1 - p^{-s}} = \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s)$$

Dedekind Zeta Function: Local Factors

For v an Archimedean place of K take $f_v(x) = e^{-\pi x^2}$ the **Gaussian function**. Then $\widehat{f_v} = f_v$ and we have

$$Z_{f_v}(s) = \int_{K_v^\times} |x|_v^s f_v(x) \mathrm{d}_v^\times x = \int_{K_v} |x|_v^{s-1} f_v(x) \mathrm{d}_v x = \Gamma_{K_v}(s)$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

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where $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. For v a finite place of K where K_v is **unramified** over $\mathbb{Q}_v$ take $f_v(x) = \chi_{\mathcal{O}_v}$ the **v-adic Gaussian function**. Then $\widehat{f_v} = f_v$ and we have

$$Z_{f_v}(s) = \int_{K_v} |x|_v^s f_v(x) \mathrm{d}_v^\times x = \int_{\mathcal{O}_v} |x|_v^s \mathrm{d}_v^\times x = \frac{1}{1 - q_v^{-s}}$$

However for v a finite place of K where K_v is **ramified** over $\mathbb{Q}_v$ there is no $f_v \in \mathcal{S}(K_v)$ with $\widehat{f_v} = f_v$.

Dedekind Zeta Function: Local Factors

For v a finite place of K where K_v is **ramified** over $\mathbb{Q}_v$ take $f_v(x) = \chi_{\mathcal{O}_v}$ the **v-adic Gaussian function**. We have

$$Z_{f_v}(s) = \frac{[\mathcal{O}_v^\times : \mathcal{O}_v]^{-\frac{1}{2}}}{1 - q_v^{-s}} \qquad Z_{\widehat{f_v}}(s) = \frac{[\mathcal{O}_v^\times : \mathcal{O}_v]^{s-\frac{1}{2}}}{1 - q_v^{-s}}$$

Now $Z_f(s) = |\Delta_K|^{-\frac{1}{2}}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s)$ and $Z_{\widehat{f}}(s) = |\Delta_K|^{s-\frac{1}{2}}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s)$.

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$$|\Delta_K|^{-\frac{1}{2}}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s) = |\Delta_K|^{(1-s)-\frac{1}{2}}\Gamma_{\mathbb{R}}(1-s)^{r_1}\Gamma_{\mathbb{C}}(1-s)^{r_2}\zeta_K(1-s)$$

Dividing by $|\Delta_K|^{-\frac{s+1}{2}}$ we obtain the functional equation

$$|\Delta_K|^{-\frac{s}{2}}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s) = |\Delta_K|^{-\frac{1-s}{2}}\Gamma_{\mathbb{R}}(1-s)^{r_1}\Gamma_{\mathbb{C}}(1-s)^{r_2}\zeta_K(1-s)$$