1. Introduction

Much of the richness of algebraic geometry comes from its beautiful interactions with, and in some cases unification of, number theory and topology. The use of these connections was one of the main themes of 20th-century number theory, and many of the great mathematical achievements of that period (for example, the proofs of the Weil conjectures and Fermat's Last Theorem) exploited the interactions between geometry, number theory, and topology.

Over the last few years, I have developed a research program aimed at using recent developments in arithmetic geometry to analyze problems at the intersection of algebraic geometry, and topology — for example, the analysis of Galois actions on fundamental groups, the geometric torsion conjecture, the Frey-Mazur conjecture for function fields, and certain aspects of archimedean and p-adic Hodge theory. Broadly speaking, the goal of this program is to use input from number theory to understand the topology of algebraic varieties, and to use input from algebraic geometry and low-dimensional topology to answer classical number-theoretic questions.

Let X be an algebraic variety, i.e. the set of solutions to a system of polynomial equations $\{f_i\}$. The main observation that lets one use arithmetic techniques to study X is that the polynomials $\{f_i\}$ are defined by a finite set of coefficients S, so one may just as well view the system of equations above as living over the finitely-generated $\mathbb{Z}$-algebra $\mathbb{Z}[S]$. Such systems of equations are accessible to the techniques of arithmetic geometry; for example, one may reduce them modulo a prime, or view them as systems of equations over a p-adic field.

The bulk of this proposal is aimed at studying the fundamental groups of algebraic varieties via these arithmetic methods. The fundamental group of X has various algebraic incarnations, in profinite, p-adic, de Rham, and other contexts. The main goal of this proposal is to develop the arithmetic theory of fundamental groups of algebraic varieties, by combining explicit ℓ-adic computations, recent developments in archimedean and p-adic Hodge theory, and analytic techniques, and to pursue applications of this theory to:

(1) the geometric torsion conjecture, the geometric Frey-Mazur conjecture, the (conjectural) Hard Lefschetz theorem for local systems in positive characteristic, and related questions about monodromy representations in algebraic geometry,
(2) questions in Hodge theory and low-dimensional topology, in particular the Putman-Wieland conjecture, and questions about mapping class groups motivated by arithmetic (for example, geometric aspects of the p-curvature conjecture and section conjecture), and
(3) the analysis of the arithmetic of rational points on algebraic varieties, particularly around the non-abelian Chabauty method and questions about rational points on K3 surfaces.

2. Anabelian geometry and monodromy

Objectives and Impact. Much of my recent work has focused on Galois actions on étale fundamental groups, and on the representation theory of arithmetic fundamental groups. Broadly speaking, the goal of this work is to compute Galois actions on fundamental groups. Beyond having fundamental intrinsic importance, these computations have applications to concrete geometric and arithmetic questions, such as the geometric torsion conjecture and the (conjectural) hard Lefschetz theorem in positive characteristic.

I anticipate that this research program will result in proofs of several well-known open conjectures, and in substantial progress in others. For example, the geometric torsion conjecture is a conjectural bound on the number of torsion points of abelian varieties over function fields; it is the function field analogue of famous results for elliptic curves over number fields due to Mazur, Merel, and others [Maz77, Mer96].

Conjecture 2.1 (Geometric torsion conjecture [BT16])**.** *Let k be an algebraically closed field. There exists an integer $N = N(g,d)$ such that if C is a smooth curve of genus g and A is a geometrically traceless d-dimensional Abelian varieties over $k(C)$, then A has at most N $k(C)$-rational torsion points.*

I believe a proof of this result is within reach; one goal of giving a proof via *anabelian* methods — the analysis of arithmetic fundamental groups — is to gain some insight into the analogous question over number fields, where essentially nothing is known except for $d = 1$.

In a somewhat different direction, I hope to prove the *Hard Lefschetz theorem* for lisse ℓ-adic sheaves on smooth proper varieties in characteristic $p \neq \ell$ (which, despite the name, is only conjectural). Recent work of Esnault and Kerz [EK21] resolves this question in rank 1 (using, among other ideas, techniques developed in [Lit18]) and suggests an approach for the general case. Resolving this question would be a fundamental advance in our understanding of the cohomology of algebraic varieties in positive characteristic.

Recent progress. By applying new dynamical techniques, [Lit18, Lit21] I've made substantial progress towards Conjecture 2.1 in characteristic zero for arbitrary g, d, and, jointly with my postdoc Borys Kadets, given the first results whatsoever towards this conjecture in positive characteristic [KL21] for $d > 1$, albeit without the desired uniformities. For example, we prove:

Theorem 2.2. *Let X be a curve over an algebraically closed field k, and let ℓ be a prime different from the characteristic of k. Let η be the function field of X. Then there exists N such that for any integer $M > N$, and any Abelian scheme $A/X_{\bar{k}}$, the following holds: if the Abelian variety A_η has full ℓ^M-torsion (that is, $A_\eta[\ell^M](\eta) = A_\eta[\ell^M](\overline{\eta})$), then A_η is isogenous to an isotrivial abelian variety over η.*

These papers also give many other consequences for the topology of algebraic varieties, many of which are new even for varieties over the complex numbers. For example, generalizing a famous result of Deligne [Del87], and giving a function field analogue of the Fontaine-Mazur conjecture [FM95, Conjectures 2a and 2b], I proved via anabelian methods:

Theorem 2.3 (L–, [Lit21])**.** *Let k be an algebraically closed field of characteristic different from ℓ, X a smooth curve over k, $\bar{x}$ a geometric point of X, and L be a finite extension of $\mathbb{Q}_\ell$. Let n be a positive integer. Then the set of isomorphism classes of tame, semisimple representations $\pi_1^{\acute{e}t}(X,\bar{x}) \to GL_n(L)$, which come from geometry, is finite.*

Literature Review. The arithmetic techniques used in [Lit18, Lit21, KL21] are essentially new, and have already sparked substantial interest from the community, for example in work of Cadoret-Moonen [CM18], who gave a proof of one of the main results of [Lit21] by different methods, and in the work of Esnault-Kerz [EK21] mentioned earlier. Most previous work on Conjecture 2.1 is analytic in nature, e.g. the work of Nadel, Hwang-To, and Bakker-Tsimerman [Nad89, HT06, BT16], and provides somewhat different uniformity.

Methodology. The approach developed in [Lit18, Lit21, KL21] ultimately boils down to an analysis of Galois actions on étale fundamental groups of algebraic varieties, and on deformation rings of residual representations of étale fundamental groups. These actions have heretofore been more or less incomputable, but I am currently developing two parallel approaches to compute them explicitly. The first approach relies on recent advances in integral p-adic Hodge theory pioneered by Bhatt-Morrow-Scholze [BMS18] to write the relevant invariants of the Galois action on the pro-p completion of the étale fundamental group of a curve over a p-adic field in terms of the valuations of certain p-adic iterated integrals in the sense of Berkovich [Ber07]. These integrals can then be directly computed, which is joint work in progress with Eric Katz, building on my previous joint work with Betts [BL19].

The second approach is more long-term and builds on the work of [KL21] to directly analyze the geometry of deformation rings $R_{\bar{\rho}}$ of residual representations of of geometric étale fundamental groups of curves over finite fields, taking input from Laurent Lafforgue's work on the Langlands program [Laf02]. Motivated in part by recent work of Pappas [Pap20], Kadets and I have already discovered a map from the rigid generic fiber of (a certain modification) of $R_{\bar{\rho}}$ to a base which loosely speaking, interpolates the Frobenius action on the tangent spaces of arithmetic representations. We conjecture that the fibers of this map are naturally torsors for certain analytic groups, by analogy to the Hitchin map — verifying this seems difficult but we expect the study of this geometric structure to be fruitful.

Training of HQP. Much of the proposed work above is joint with my postdoc Borys Kadets. The computational aspects of p-adic iterated integrals are suitable for a masters or PhD student; in particular, they lead to explicit algorithms which would be extremely useful to implement on a computer. The implementation of these algorithms would lead to (1) explicit computations of p-adic quantities of arithmetic interest, e.g. p-adic multiple zeta values, and (2) new algorithms for finding rational points on curves, for example via an implementation of the non-abelian Chabauty method on curves of bad reduction.

3. The arithmetic and geometry of mapping class groups

Objectives and Impact. It is often useful to consider specific monodromy representations—of particular interest is the outer action of the mapping class group of a surface (the fundamental group of the moduli space of genus g curves with n marked points, $\mathscr{M}_{g,n}$) on the fundamental group of a surface of genus g with n punctures. This action may be studied both via arithmetic and algebraic geometry, due to its interpretation in terms of algebraic geometry: it is encoded in the geometry of the forgetful morphism $\mathscr{M}_{g,n+1} \to \mathscr{M}_g$. Let $\Sigma_{g,n}$ be a smooth orientable (topological) surface of genus g, with n punctures. The outer action of the mapping class group of $\Sigma_{g,n}$ on $\pi_1(\Sigma_{g,n})$ induces an action on the set of isomorphism classes of representations of $\pi_1(\Sigma_{g,n})$ — understanding this action is fundamental to several of the most important open questions in algebraic geometry, number theory, and low-dimensional topology. I plan to take advantage of the algebro-geometric nature of this action to study a constellation of questions motivated by low-dimensional topology (in particular, the Putman-Wieland [PW13] conjecture and Ivanov's conjecture on mapping class groups) and arithmetic (for example, questions around the Grothendieck-Katz p-curvature conjecture and Grothendieck's section conjecture).

Let $\rho : \pi_1(\Sigma_{g,n}) \to GL_n(\mathbb{C})$ be a representation.

Conjecture 3.1 (Putman-Wieland)**.** *If $g > 2$ and ρ has finite image, the (virtual) action of the mapping class group of $\Sigma_{g,n}$ on the group cohomology $H^1(\pi_1(\Sigma_{g,n}), \rho)$ has no non-zero vectors with finite orbit.*

This conjecture is of fundamental importance in algebraic geometry — it is equivalent to the non-existence of isotrivial factors of higher Prym varieties — and low-dimensional topology, where it is known to imply that finite index subgroups of mapping class groups of surfaces of large genus have finite abelianization, a well-known conjecture of Ivanov [PW13].

Likewise, the Grothendieck-Katz p-curvature conjecture may be reformulated in terms of this mapping class group action [LL19]. Indeed, it is equivalent to the statement that if ρ is the representation associated to a flat vector bundle on a curve, almost all of whose p-curvatures vanish, then the mapping class group orbit of ρ is finite.

I propose to prove the Putman-Wieland conjecture, and I hope to make progress on the Grothendieck-Katz p-curvature conjecture from this point of view.

And in recent joint work with Li, Salter, and Srinivasan [LLSS20], I have conjectured geometric analogues of Grothendieck's section conjecture (involving an analysis of these mapping class group actions) called the *tropical section conjecture* which lead to arithmetic cases where one may verify the section conjecture; I plan to prove these geometric analogues.

Recent progress. In ongoing work with Aaron Landesman (building on [LL19]), I have proven via new Hodge-theoretic techniques:

Theorem 3.2 (Landesman-Litt)**.** *The Putman-Wieland conjecture holds if* $g \geq \dim \rho$.

In the short term, we think it is plausible that these methods can be extended to prove the entire conjecture and shed light on related questions: for example, the approach to the p-curvature conjecture laid out in [LL19]. In the long term, building on [PSW21], we hope to prove the p-curvature conjecture for vector bundles of rank r on the generic curve of genus g for $g \gg r$. With regards to the section conjecture, the paper [LLSS20] verifies many cases of the tropical section conjecture, and a full proof seems to be in reach.

Literature Review. All three areas discussed here—the Putman-Wieland conjecture, the p-curvature conjecture, and the section conjecture—are extremely active areas of research. With regards to the Putman-Wieland conjecture, my ongoing work with Landesman is the first Hodge-theoretic approach to the conjecture, but notable existing partial results on the conjecture include work of Grunewald, Larsen, Lubotzky, Malestein, and Looijenga [GLLM15, Loo21], which uses more classical topological techniques. The proposed work on the p-curvature conjecture is motivated by a folklore question of Kisin and work of Shankar [Sha18] and Shankar-Patel-Whang [PSW21]. And our geometric analogue of the section conjecture is essentially new but heavily influenced by work of Hain [Hai11].

Methodology. The main insight here is that one may bring tools from Hodge theory (and p-adic Hodge theory) to bear on all the questions discussed in this section. Due to space constraints, I will focus only here only on the Putman-Wieland conjecture. Here, we observe that the failure of the Putman-Wieland conjecture implies that a certain variation of Hodge structure on $\mathscr{M}_{g,n}$ has Kodaira-Spencer map of lower-than-expected rank. An infinitesimal analysis yields a construction of a stable vector bundle of high degree which is not generically globally generated; this yields a contradiction in the range given in Theorem 3.2.

Training of HQP. A number of questions around those raised here are suitable for doctoral dissertations, in particular questions around the tropical section conjecture. While this conjecture can be approached via e.g. Hodge-theoretic techniques, it is also amenable to direct computation. I also have in mind a generalization relative to a fixed representation of $\pi_1(\Sigma_{g,n})$, which promises to open up new structure theory of the mapping class group, for example a generalization of the Johnson homomorphism to certain interesting subgroups of the mapping class group which are as yet understudied.

4. ARITHMETIC OF VARIETIES WITH BAD REDUCTION

Objectives and Impact. Recent developments in the arithmetic of fundamental groups—in particular, the non-abelian Chabauty method for finding rational points on curves and its variants—have proven extremely important in answering diophantine questions. For example, Balakrishnan et al [BDM$^+$19] were able to use these methods to find the rational points on the so-called "cursed curve." These methods require a choice of prime where the curve has good reduction — this choice is (1) an obstruction to using these methods to produce uniform bounds on the number of rational points on a curve (since the curve may have bad reduction at all small primes), and (2) an obstruction to rapid computation, since the complexity of the computation grows rapidly with the prime in question. Thus it is desirable to modify these techniques to work in the bad reduction setting.

Ongoing joint work with Eric Katz (building on my joint work with Betts [BL19], and mentioned also in §2) will provide one key ingredient of this — namely, explicit methods to compute so-called Berkovich and Vologodsky iterated integrals on curves with bad reduction over p-adic fields. Short-term, this is essentially the last necessary ingredient necessary to run the non-abelian Chabauty method in bad reduction.

Long-term, this work is part of an ongoing program to better understand the arithmetic and geometry of algebraic varieties with "totally degenerate" reduction, which promises to have many applications. For example, in ongoing work with Philip Engel, I will use related methods to analyze the arithmetic of rational curves on K3 surfaces with very bad reduction, with the aim of answering the following question of Bogomolov-Tschinkel [BT00] in the negative: is there a rational curve through every rational point on a K3 surface over $\overline{\mathbb{Q}}$? Beyond this, I expect that related methods will give access to questions about algebraic cycles which are inaccessible with current techniques — for example, if a variety over $\mathbb{C}((t))$ has bad reduction, one may use related techniques to detect cycles in the third piece of the (conjectural) Bloch-Beilinson filtration, which is not detected by existing Hodge-theoretic methods.

Recent progress. The theoretical component of my joint work with Katz underpinning the explicit computation of Berkovich and Vologodsky integrals is complete, and should be available on arXiv in the coming weeks. And building on techniques developed in [Lit18], I have proven in unpublished joint work with Engel, Krishnamoorthy, and Shankar, the following partial result towards Bogomolov-Tschinkel's question:

Theorem 4.1. *Let $X/\overline{\mathbb{Q}}$ be a Kummer K3 surface associated to a simple Abelian surface A. Then there exist infinitely many $\overline{\mathbb{Q}}$-points of X not contained in a rational curve whose preimage in A has geometric genus* 2.

This is in contrast to the situation over the algebraic closure of a finite field, where Bogomolov-Tschinkel prove that such curves pass through every rational point — this is what motivated their original question.

Literature Review. The proposed work on iterated integrals is the p-adic part of a program whose ℓ-adic component (for $\ell \neq p$) was completed by Betts-Dogra [BD19]. Explicit algorithms in some special situations (e.g. for hyperelliptic curves) have been developed by Kaya and Kaya-Katz [Kay20, KK20]. There has been much less progress on the other problems mentioned here — particularly questions on the arithmetic of rational curves on K3 surfaces, with the notable exception of [BM10], which proves some partial results.

Methodology. The approach here is a mixture of tropical, p-adic Hodge theoretic, and arithmetic techniques (in part motivated by those in [BLLS20]). Loosely speaking the cohomology or fundamental group of a variety with sufficiently bad reduction may be broken into a geometric component (arising from the geometry of the components of the special fiber) and a combinatorial component (arising from its dual complex). These components are typically more tractable than the original questions about the generic fiber.

Training of HQP. There are many algorithmic questions around iterated integrals (also mentioned in §2), and implementing these on a computer would be an incredibly useful project for a PhD student. More broadly, these sorts of questions around arithmetic of varieties of bad reduction admit p-adic, Hodge-theoretic, and tropical aspects, each of which is suitable for a dissertation; some of the combinatorial aspects will also make interesting projects for masters students.

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