Fourier Theory on $\mathbb{R}$ and $\mathbb{Q}_p$

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Motivation: Riemann Zeta Function

Consider the Riemann zeta function

$$\zeta(s) = \sum_{\substack{n \subseteq \mathbb{Z} \\ \text{ideal}}} \frac{1}{n^s} = \prod_{\substack{p \subseteq \mathbb{Z} \\ \text{prime}}} \frac{1}{1 - p^{-s}} \qquad \Re(s) > 1$$

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Theorem

The completed zeta function $Z(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ admits an **analytic continuation** with a simple pole at $s = 1$ with residue 1, and satisfies the **functional equation**

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\tfrac{1-s}{2})\zeta(1-s)$$

We want to understand the **local factors** $\zeta_\infty(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $\zeta_p(s) = \frac{1}{1-p^{-s}}$ as integrals over $\mathbb{R}$ and $\mathbb{Q}_p$.

Fourier Theory over $\mathbb{R}$

Let $\mathcal{S}(\mathbb{R})$ denote the space of Schartz functions on $\mathbb{R}$: the $\mathbb{C}$-vector space of smooth functions $f : \mathbb{R} \to \mathbb{C}$ such that $f^{(n)} : \mathbb{R} \to \mathbb{C}$ has rapid decay for all $n \geq 0$.
The field $\mathbb{R}$ is a locally compact Abelian group which is **Pontryagin self-dual**, $\widehat{\mathbb{R}} = \mathbb{R}$.

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Definition

For $f_\infty \in \mathcal{S}(\mathbb{R})$ define the Fourier transform

$$\widehat{f_\infty}(u) = \int_{\mathbb{R}} f_\infty(x)\chi_{\infty,u}(x)\mathrm{d}x = \int_{\mathbb{R}} f_\infty(x)e^{-2\pi iux}\mathrm{d}x$$

Define the inverse Fourier transform

$$f_\infty(x) = \int_{\mathbb{R}} \widehat{f_\infty}(u)\overline{\chi_{\infty,u}(x)}\mathrm{d}u = \int_{\mathbb{R}} \widehat{f_\infty}(u)e^{2\pi iux}\mathrm{d}u$$

Fourier Theory over $\mathbb{R}$

Example (Gaussian Function)

Let $f_\infty(x) = e^{-cx^2}$ be the Gaussian function. Then

$$\begin{aligned}
\widehat{f_\infty}(u) &= \int_{\mathbb{R}} f_\infty(x)\chi_{\infty,u}(x)\mathrm{d}x = \int_{\mathbb{R}} e^{-cx^2} e^{-2\pi iux}\mathrm{d}x \\
&= \int_{\mathbb{R}} e^{-cx^2}\cos(2\pi ux)\mathrm{d}x - i\int_{\mathbb{R}} e^{-cx^2}\sin(2\pi ux)\mathrm{d}x \\
&= \int_{\mathbb{R}} e^{-cx^2}\cos(2\pi ux)\mathrm{d}x = \sqrt{\frac{\pi}{c}}e^{-\frac{\pi^2u^2}{c}}
\end{aligned}$$

Hence the **Gaussian function** $f_\infty(x) = e^{-\pi x^2}$ is **Fourier self-dual**, $\widehat{f_\infty} = f_\infty$.

Fourier Theory over $\mathbb{R}$

Fix the Haar measure $\mathrm{d}^\times x$ on $\mathbb{R}^\times$. We have $\mathrm{d}^\times x = \frac{\mathrm{d}x}{|x|}$.

Example

(Local factor $\zeta_\infty(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$) Let $f_\infty(x) = e^{-\pi x^2}$ be the **Gaussian function**. Then $\widehat{f_\infty} = f_\infty$, and we have

$$Z_{f_\infty}(s) = \int_{\mathbb{R}^\times} |x|^s f_\infty(x)\mathrm{d}^\times x = \int_{\mathbb{R}} |x|^{s-1} f_\infty(x)\mathrm{d}x = \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})$$

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For this use the integral definition of the Gamma function

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\mathrm{d}x$$

so indeed $Z_{f_\infty}(s) = \zeta_\infty(s)$.

Fourier Theory over $\mathbb{R}$

Example (Bessel Function)

Consider the modified Bessel functions

$$I_s(x) = \sum_{n\geq 0} \frac{1}{n!\Gamma(n+s+1)}\left(\frac{x}{2}\right)^{2n+s} \qquad K_s(x) = \frac{\pi}{2}\frac{I_{-s}(x) - I_s(x)}{\sin(sx)}$$

$I_s(x)$ and $K_s(x)$ are the two linearly independent solutions to the modified Bessel equation

$$x^2\frac{\mathrm{d}^2 f_s}{\mathrm{d}x^2} + x\frac{\mathrm{d}f_s}{\mathrm{d}x} - (x^2+s^2)f_s = 0$$

The modified **Bessel function** $K_s(x)$ can be written as an inverse Fourier transform of $||(1,u)||^{-2s} = (1+u^2)^{-s}$ by

$$\int_{\mathbb{R}} (1+u^2)^{-s} e^{-2\pi i u x}\mathrm{d}u = \frac{2\pi^s}{\Gamma(s)}|x|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|x|)$$

Integration over $\mathbb{Q}_p$

Fix a Haar measure $\mathrm{d}x$ on $\mathbb{Q}_p$ normalized so that $\int_{\mathbb{Z}_p} \mathrm{d}x = 1$. We use the decompositions $\mathbb{Z}_p = \coprod_{0 \leq k \leq p-1} Z_k$ where $Z_k = \{x \in \mathbb{Z}_p \mid x = k + \mathrm{O}(p)\}$, and $\mathbb{Z}_p = \coprod_{k \geq 0} p^k \mathbb{Z}_p^\times$.

Example

$$\int_{\mathbb{Z}_p^\times} \mathrm{d}x = \frac{p-1}{p}$$

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Example

$$\int_{\mathbb{Z}_p^\times} \mathrm{d}x = \frac{p-1}{p}$$

For this use $\mathbb{Z}_p = \coprod_{0 \le k \le p-1} Z_k$ so

$$\int_{\mathbb{Z}_p^\times} \mathrm{d}x = \sum_{1 \le k \le p-1} \int_{Z_k} \mathrm{d}x = \sum_{1 \le k \le p-1} \frac{1}{p} = \frac{p-1}{p}$$

Integration over $\mathbb{Q}_p$

Example

$$\int_{\mathbb{Z}_p} |x|_p^s \mathrm{d}x = \frac{p-1}{p}\frac{1}{1-p^{-s-1}} \qquad \Re(s) > -1$$

Integration over $\mathbb{Q}_p$

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$$\int_{\mathbb{Z}_p} |x|_p^s \mathrm{d}x = \frac{p-1}{p} \frac{1}{1-p^{-s-1}} \qquad \Re(s) > -1$$

For this use $\mathbb{Z}_p = \coprod_{k\geq 0} p^k \mathbb{Z}_p^\times$ and change variables $x = p^k u$ so

$$\int_{\mathbb{Z}_p} |x|_p^s \mathrm{d}x = \sum_{k\geq 0} \int_{p^k\mathbb{Z}_p^\times} |x|_p^s \mathrm{d}x = \sum_{k\geq 0} p^{-ks} \int_{p^k\mathbb{Z}_p^\times} \mathrm{d}x$$

$$= \sum_{k\geq 0} p^{-ks} \int_{\mathbb{Z}_p^\times} p^{-k} \mathrm{d}u = \frac{p-1}{p} \sum_{k\geq 0} p^{-k(s+1)} = \frac{p-1}{p} \frac{1}{1-p^{-s-1}}$$

which converges for $\Re(s) > -1$.

Integration over $\mathbb{Q}_p$

Example

$$\int_{\mathbb{Q}_p - \mathbb{Z}_p} |x|_p^s \mathrm{d}x = \frac{p-1}{p} \frac{p^{s+1}}{1-p^{s+1}} \qquad \Re(s) < -1$$

Integration over $\mathbb{Q}_p$

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For this we use $\mathbb{Q}_p - \mathbb{Z}_p = \coprod_{k \geq 1} p^{-k}\mathbb{Z}_p^\times$ so

$$\int_{\mathbb{Q}_p - \mathbb{Z}_p} |x|_p^s \mathrm{d}x = \sum_{k \geq 1} p^{ks} \int_{p^{-k}\mathbb{Z}_p^\times} \mathrm{d}x = \sum_{k \geq 1} p^{k(s+1)} \int_{\mathbb{Z}_p^\times} \mathrm{d}x$$

$$= \frac{p-1}{p} \sum_{k \geq 1} p^{k(s+1)} = \frac{p-1}{p} \frac{p^{s+1}}{1-p^{s+1}}$$

which converges for $\Re(s) < -1$. **Note that the same integral over $\mathbb{Q}_p$ does not exist!**

Integration over $\mathbb{Q}_p$

For $x \in \mathbb{Q}_p$ let $[x]_p$ denote the fractional part of x, that is

$$[x_k p^k + \ldots + x_{-1} p^{-1} + x_0 p^0 + x_1 p^1 + \ldots]_p = \begin{cases} x_k p^k + \ldots + x_{-1} p^{-1} & k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that for $x \in \mathbb{Q}$, $x - \sum_p [x]_p \in \mathbb{Z}$.

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Note that for $x \in \mathbb{Q}$, $x - \sum_p [x]_p \in \mathbb{Z}$.
For $u \in \mathbb{Q}_p$ consider the additive character $\chi_{p,u} : \mathbb{Q}_p \to \mathrm{U}(1)$ given for $x \in \mathbb{Q}_p$ by $\chi_{p,u}(x) = e^{-2\pi i [ux]_p}$. The **conductor** of $\chi_{p,u}$ is the kernel $|u|_p \mathbb{Z}_p \subseteq \mathbb{Q}_p$. Then $\chi_{p,u}(x)$ is additive in $u, x \in \mathbb{Q}_p$ and satisfies $\overline{\chi_{p,u}(x)} = \chi_{p,-u}(x) = \chi_{p,u}(-x)$.

Integration over $\mathbb{Q}_p$

Example

$$\int_{p^k\mathbb{Z}_p} \chi_{p,u}(x)\mathrm{d}x = \int_{p^k\mathbb{Z}_p} e^{-2\pi i[ux]_p}\mathrm{d}x = p^{-k}\gamma_p(up^k) \qquad k \in \mathbb{Z}$$

where $\gamma_p(x)$ is the **p-adic Gaussian** defined by

$$\gamma_p(u) = \int_{\mathbb{Z}_p} \chi_{p,u}(x)\mathrm{d}x = \int_{\mathbb{Z}_p} e^{-2\pi i[ux]_p}\mathrm{d}x = \begin{cases} 1 & u \in \mathbb{Z}_p \\ 0 & u \notin \mathbb{Z}_p \end{cases}$$

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For $k = 0$ the integral depends only on the conductor $|u|_p\mathbb{Z}_p$. For $k \neq 0$ we change variables so

$$\int_{p^k\mathbb{Z}_p} \chi_{p,u}(x)\mathrm{d}x = \int_{p^k\mathbb{Z}_p} e^{-2\pi i[ux]_p}\mathrm{d}x = p^{-k}\int_{\mathbb{Z}_p} e^{-2\pi i[up^k x]_p}\mathrm{d}x = p^{-k}\gamma_p(up^k)$$

Integration over $\mathbb{Q}_p$

Example

$$\int_{p^k\mathbb{Z}_p^\times} \chi_{p,u}(x)\mathrm{d}x = \int_{p^k\mathbb{Z}_p^\times} e^{-2\pi i[ux]_p}\mathrm{d}x = \begin{cases} \frac{p-1}{p}p^{-k} & |u|_p \leq p^k \\ -p^{-(k+1)} & |u|_p = p^{k+1} \\ 0 & |u|_p > p^{k+1} \end{cases} \qquad k \in \mathbb{Z}$$

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For $k = 0$ we use $\mathbb{Z}_p^\times = \mathbb{Z}_p - p\mathbb{Z}_p$ so

$$\int_{\mathbb{Z}_p^\times} \chi_{p,u}(x)\mathrm{d}x = \int_{\mathbb{Z}_p^\times} e^{-2\pi i[ux]_p}\mathrm{d}x = \int_{\mathbb{Z}_p} e^{-2\pi i[ux]_p}\mathrm{d}x - \int_{p\mathbb{Z}_p} e^{-2\pi i[ux]_p}\mathrm{d}x$$

$$= \gamma_p(u) - p^{-1}\int_{\mathbb{Z}_p} e^{-2\pi i[upx]_p}\mathrm{d}x = \gamma_p(u) - p^{-1}\gamma_p(up) = \begin{cases} \frac{p-1}{p} & |u|_p \leq 1 \\ -p^{-1} & |u|_p = p \\ 0 & |u|_p > p \end{cases}$$

Integration over $\mathbb{Q}_p$

For $k \neq 0$ we change variables so

$$\int_{p^k\mathbb{Z}_p^\times} \chi_{p,u}(x)\mathrm{d}x = \int_{p^k\mathbb{Z}_p^\times} e^{-2\pi i[ux]_p}\mathrm{d}x = p^{-k}\int_{\mathbb{Z}_p^\times} e^{-2\pi i[up^k x]_p}\mathrm{d}x$$

$$= p^{-k}\gamma_p(up^k) - p^{-(k+1)}\gamma_p(up^{k+1}) = \begin{cases} \frac{p-1}{p}p^{-k} & |u|_p \leq p^k \\ -p^{-(k+1)} & |u|_p = p^{k+1} \\ 0 & |u|_p > p^{k+1} \end{cases}$$

Fourier Theory over $\mathbb{Q}_p$

Let $\mathcal{S}(\mathbb{Q}_p)$ denote the space of Bruhat-Schwartz functions on $\mathbb{Q}_p$: the $\mathbb{C}$-vector space of compactly supported locally constant functions $f : \mathbb{Q}_p \to \mathbb{C}$.
The field $\mathbb{Q}_p$ is a locally compact Abelian group which is **Pontryagin self-dual**, $\widehat{\mathbb{Q}}_p = \mathbb{Q}_p$.

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Definition

For $f_p \in \mathcal{S}(\mathbb{Q}_p)$ define the Fourier transform

$$\widehat{f_p}(u) = \int_{\mathbb{Q}_p} f_p(x)\chi_{p,u}(x)\mathrm{d}x = \int_{\mathbb{Q}_p} f_p(x)e^{-2\pi i[ux]_p}\mathrm{d}x$$

Define the inverse Fourier transform

$$f_p(x) = \int_{\mathbb{Q}_p} \widehat{f_p}(u)\overline{\chi_{p,u}(x)}\mathrm{d}u = \int_{\mathbb{Q}_p} \widehat{f_p}(u)e^{2\pi i[ux]_p}\mathrm{d}u$$

Fourier Theory over $\mathbb{Q}_p$

Recall by a previous example,

$$\int_{p^k\mathbb{Z}_p} \chi_{p,u}(x)\mathrm{d}x = \int_{p^k\mathbb{Z}_p} e^{-2\pi i [ux]_p}\mathrm{d}x = p^{-k}\gamma_p(up^k) \qquad k \in \mathbb{Z}$$

Example (p-adic Gaussian Function)

Let $f_p(x) = \gamma_p(x)$ be the **p-adic Gaussian function**. Then

$$\widehat{f_p}(u) = \int_{\mathbb{Q}_p} f_p(x)\chi_{p,u}(x)\mathrm{d}x = \int_{\mathbb{Z}_p} \chi_{p,u}(x)\mathrm{d}x$$
$$= \int_{\mathbb{Z}_p} e^{-2\pi i [ux]_p}\mathrm{d}x = f_p(u)$$

by a previous example. Hence the **p-adic Gaussian function** $f_p(x) = \gamma_p(x)$ is **Fourier self-dual**, $\widehat{f_p} = f_p$.

Fourier Theory over $\mathbb{Q}_p$

Example

$$\int_{\mathbb{Q}_p-\mathbb{Z}_p} |x|_p^s \chi_{p,u}(x) \mathrm{d}x = \gamma_p(u)\left((1-p^s)\frac{1-p^{s+1}|u|_p^{-s-1}}{1-p^{s+1}} - 1\right)$$

Fourier Theory over $\mathbb{Q}_p$

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The integral depends only on the conductor $|u|_p \mathbb{Z}_p$. If $u \in \mathbb{Z}_p$ with conductor p^k for $k \geq 0$ we have

$$\int_{\mathbb{Q}_p - \mathbb{Z}_p} |x|_p^s \chi_{p,u}(x) \mathrm{d}x = \int_{\mathbb{Q}_p - \mathbb{Z}_p} |x|_p^s e^{-2\pi i [p^k x]_p} \mathrm{d}x = \sum_{\ell \geq 1} p^{s\ell} \int_{p^{-\ell}\mathbb{Z}_p^\times} e^{-2\pi i [p^k x]_p} \mathrm{d}x$$

$$= \sum_{\ell \geq 1} p^{(s+1)\ell} \int_{\mathbb{Z}_p^\times} e^{-2\pi i [p^{k-\ell} x]_p} \mathrm{d}x = \frac{p-1}{p} \sum_{1 \leq \ell \leq k} p^{(s+1)\ell} - \frac{1}{p} p^{(k+1)(s+1)}$$

$$= (1-p^s)\frac{1-p^{s+1}|u|_p^{-s-1}}{1-p^{s+1}} - 1$$

If $u \notin \mathbb{Z}_p$ with conductor p^k for $k < 0$ the above integral vanishes.

Fourier Theory over $\mathbb{Q}_p$

Fix a Haar measure $\mathrm{d}^\times x$ on $\mathbb{Q}_p^\times$ normalized so that $\int_{\mathbb{Z}_p^\times} \mathrm{d}^\times x = 1$. We have $\mathrm{d}^\times x = \frac{p}{p-1} \frac{\mathrm{d}x}{|x|_p}$.

Example

(Local factor $\zeta_p(s) = \frac{1}{1-p^{-s}}$) Let $f_p(x) = \gamma_p(x)$ be the **p-adic Gaussian function**. Then $\widehat{f_p} = f_p$ and we have

$$Z_{f_p}(s) = \int_{\mathbb{Q}_p} |x|_p^s f_p(x) \mathrm{d}^\times x = \int_{\mathbb{Z}_p} |x|_p^s \mathrm{d}^\times x = \frac{1}{1-p^{-s}} \qquad s \neq 0$$

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For this we use $\mathbb{Z}_p = \coprod_{k \geq 0} p^k \mathbb{Z}_p^\times$ so

$$\int_{\mathbb{Z}_p} |x|_p^s \mathrm{d}^\times x = \sum_{k \geq 0} p^{-ks} \int_{\mathbb{Z}_p^\times} \mathrm{d}^\times x = \sum_{k \geq 0} p^{-ks} = \frac{1}{1-p^{-s}} \qquad s \neq 0$$

Fourier Theory over $\mathbb{Q}_p$

Example (p-adic Bessel Function)

The modified **p-adic Bessel function** $K_{p,s}(x)$ can be written as an inverse Fourier transform of $||(1,u)||^{-2s} = \max(1,|u|_p)^{-2s}$, normalized by $\frac{1}{1-p^{-2s}}$ by

$$\frac{1}{1-p^{-2s}}\int_{\mathbb{Q}_p} \max(1,|u|_p)^{-2s} e^{2\pi i[ux]_p} \mathrm{d}u = \gamma_p(x)\frac{1-p^{-2s+1}|x|_p^{2s-1}}{1-p^{-2s+1}}$$

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By previous examples,

$$\begin{aligned}
&\frac{1}{1-p^{-2s}}\int_{\mathbb{Q}_p} \max(1,|u|_p)^{-2s} e^{2\pi i[ux]_p}\mathrm{d}u \\
&= \frac{1}{1-p^{-2s}}\int_{\mathbb{Z}_p} e^{2\pi i[ux]_p}\mathrm{d}u + \frac{1}{1-p^{-2s}}\int_{\mathbb{Q}_p-\mathbb{Z}_p} |u|_p^{-2s} e^{2\pi i[ux]_p}\mathrm{d}u \\
&= \frac{\gamma_p(x)}{1-p^{-2s}} + \frac{\gamma_p(x)}{1-p^{-2s}}\left((1-p^{-2s})\frac{1-p^{-2s+1}|u|_p^{2s-1}}{1-p^{-2s+1}} - 1\right) = \gamma_p(x)\frac{1-p^{-2s+1}|x|_p^{2s-1}}{1-p^{-2s+1}}
\end{aligned}$$

Fourier Theory over $\mathbb{A}_{\mathbb{Q}}$

Let $\mathcal{S}(\mathbb{A}_{\mathbb{Q}})$ denote the space of Bruhat-Schwarz functions on $\mathbb{A}_{\mathbb{Q}}$: the $\mathbb{C}$-vector space of finite $\mathbb{C}$-linear combinations of monomial Schwartz functions $f : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$, namely $f(x) = \prod_v f_v(x_v)$ for $f_v = \gamma_p$ for all but finitely many finite places v of $\mathbb{Q}$.

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Definition

For $f \in \mathcal{S}(\mathbb{A}_{\mathbb{Q}})$ define the Fourier transform

$$\widehat{f}(u) = \int_{\mathbb{A}_{\mathbb{Q}}} f(x)\chi_u(x)\mathrm{d}x = \int_{\mathbb{A}_{\mathbb{Q}}} f(x)e^{-2\pi i[ux]}\mathrm{d}x$$

Define the inverse Fourier transform

$$f(x) = \int_{\mathbb{A}_{\mathbb{Q}}} \widehat{f}(u)\overline{\chi_u(x)}\mathrm{d}u = \int_{\mathbb{A}_{\mathbb{Q}}} \widehat{f}(u)e^{2\pi i[ux]}\mathrm{d}u$$

For $f = \prod_v f_v$ a monomial Schwartz function we have $\widehat{f} = \prod_v \widehat{f_v}$.

Zeta Integrals over $\mathbb{A}_{\mathbb{Q}}$

Definition

For $f \in \mathcal{S}(\mathbb{A}_{\mathbb{Q}})$ define the global zeta integral

$$Z_f(s) = \int_{\mathbb{I}_{\mathbb{Q}}} |x|^s f(x) \mathrm{d}^{\times} x$$

For $f = \prod_v f_v$ a monomial Schwartz function we have $Z_f(s) = \prod_v Z_{f_v}(s)$ for $\Re(s) > 1$.

Zeta Integrals over $\mathbb{A}_{\mathbb{Q}}$

Definition

For $f \in \mathcal{S}(\mathbb{A}_{\mathbb{Q}})$ define the global zeta integral

$$Z_f(s) = \int_{\mathbb{I}_{\mathbb{Q}}} |x|^s f(x) \mathrm{d}^\times x$$

For $f = \prod_v f_v$ a monomial Schwartz function we have $Z_f(s) = \prod_v Z_{f_v}(s)$ for $\Re(s) > 1$.

Example (Completed Riemann Zeta Function)

Let $f = \prod_v \gamma_v$ be the **global Gaussian function**. Then $\widehat{f} = f$, and

$$Z_f(s) = \prod_v Z_{f_v}(s) = \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \prod_p \frac{1}{1 - p^{-s}}$$

so indeed $Z_f(s) = Z(s)$.

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Example (Non-Holomorphic Eisenstein Series)

The non-holomorphic Eisenstein series $E_s(z)$ has Fourier expansion $E_s(z) = \sum_{n\in\mathbb{Z}} E_s(y)_n e^{2\pi i n x}$. Its coefficients are given

$$E_s(z) = y^s + \frac{Z(2s-1)}{Z(2s)} y^{1-s} + \sum_{n\in\mathbb{Z}} \frac{2}{Z(2s)} |n|^{s-\frac{1}{2}} y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y)\sigma_{1-2s}(n) e^{2\pi i n x}$$

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The $\frac{2}{Z(2s)}|n|^{s-\frac{1}{2}} y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y) = \frac{1}{\zeta(2s)}\frac{2\pi^s}{\Gamma(s)}|n|^{s-\frac{1}{2}} y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y)$ is coming from the **Bessel function**. The generalized divisor sum $\sigma_{1-2s}(n)$ is coming from the **p-adic Bessel functions**:

$$\prod_p \left(\gamma_p(x)\frac{1-p^{-2s+1}|n|_p^{2s-1}}{1-p^{-2s+1}}\right) = \sum_{d|n} d^{1-2s} = \sigma_{1-2s}(n)$$

Indeed $E_s(z)$ can be viewed as a certain integral over $\mathbb{A}_{\mathbb{Q}}$.