A Frobenius-persistent nonvanishing for an ample bundle on $\mathrm{Gr}(2,4)$

Statement

Let k be a perfect field of characteristic $p > 0$, and let $F = F_X$ denote the absolute Frobenius of a k-scheme X. The purpose of this note is to give a self-contained counterexample to the following plausible vanishing assertion:

$$H^i\big(X, F^{n*}\mathscr{E} \otimes \mathscr{G}\big) = 0 \quad \text{for all } i \geq \mathrm{rk}(\mathscr{E}) \text{ and all } n \gg 0,$$

where X is smooth projective, $\mathscr{E}$ is ample, and $\mathscr{G}$ is coherent.

Theorem 1. *Let k be a perfect field of characteristic $p > 0$. There exist a smooth projective Grassmannian $X = \mathrm{Gr}(2,4)$, an ample vector bundle $\mathscr{E}$ of rank 2 on X, and a coherent sheaf $\mathscr{G}$ on X such that*

$$H^2\big(X, F^{n*}\mathscr{E} \otimes \mathscr{G}\big) \neq 0 \quad \textit{for every } n \geq 0.$$

In particular, Frobenius pullbacks of ample vector bundles need not eventually kill cohomology in degrees $i \geq \mathrm{rk}(\mathscr{E})$.

The example will be

$$X = \mathrm{Gr}(2, k^4), \qquad \mathscr{E} = F^*\mathscr{Q} \otimes \mathcal{O}_X(1), \qquad \mathscr{G} = \omega_X,$$

where

$$0 \longrightarrow \mathscr{S} \longrightarrow k^4 \otimes \mathcal{O}_X \longrightarrow \mathscr{Q} \longrightarrow 0$$

is the tautological sequence and $\mathcal{O}_X(1) = \det \mathscr{Q}$.

1 Persistence of nonvanishing on Frobenius-split varieties

We first isolate the mechanism that makes a single nonvanishing persist under all Frobenius pullbacks.

Lemma 2 (Persistence lemma)**.** *Let Y be a smooth projective Frobenius-split variety over k. Let $\mathscr{A}$ be a vector bundle on Y. If*

$$H^i(Y, \mathscr{A} \otimes \omega_Y) \neq 0,$$

then

$$H^i\big(Y, F^{n*}\mathscr{A} \otimes \omega_Y\big) \neq 0 \quad \textit{for every } n \geq 0.$$

More precisely, after restricting scalars along the n-fold Frobenius of k,

$$H^i(Y, \mathscr{A} \otimes \omega_Y)$$

is a direct summand of

$$H^i\big(Y, F^{n*}\mathscr{A} \otimes \omega_Y\big).$$

Proof. Since Y is Frobenius split, the natural inclusion

$$\mathcal{O}_Y \longrightarrow F_*\mathcal{O}_Y$$

splits as a map of $\mathcal{O}_Y$-modules. Iterating the splitting gives, for every $n \geq 0$, a decomposition

$$F_*^n\mathcal{O}_Y \simeq \mathcal{O}_Y \oplus \mathscr{R}_n$$

for some $\mathcal{O}_Y$-module $\mathscr{R}_n$.

Because Y is smooth, Grothendieck duality for the finite flat map $F^n : Y \to Y$ gives

$$\mathcal{H}om_{\mathcal{O}_Y}\big(F_*^n\mathcal{O}_Y, \omega_Y\big) \simeq F_*^n\omega_Y.$$

Equivalently, this is the n-fold Cartier trace pairing. Etale-locally, if $t_1, \ldots, t_d$ are parameters on Y, then the local basis elements

$$F_*^n t^\alpha, \qquad 0 \leq \alpha_j < p^n,$$

of $F_*^n\mathcal{O}_Y$ pair perfectly with the local basis elements

$$F_*^n\big(t^{(p^n-1,\ldots,p^n-1)-\alpha}\, dt_1 \wedge \cdots \wedge dt_d\big)$$

of $F_*^n\omega_Y$.

Applying $\mathcal{H}om_{\mathcal{O}_Y}(-, \omega_Y)$ to the direct-sum decomposition of $F_*^n\mathcal{O}_Y$, we see that ω_Y is a direct summand of $F_*^n\omega_Y$. Tensoring with $\mathscr{A}$ and using the projection formula gives $\mathscr{A} \otimes \omega_Y$ as a direct summand of

$$\mathscr{A} \otimes F_*^n\omega_Y \simeq F_*^n\big(F^{n*}\mathscr{A} \otimes \omega_Y\big).$$

Taking cohomology proves the direct-summand assertion. The Frobenius twist of the k-vector-space structure is irrelevant for vanishing, so the stated nonvanishing follows. □

2 The Grassmannian and its Frobenius splitting

Let $V = k^4$ and choose a volume form $\det V \simeq k$. Let

$$X = \mathrm{Gr}(2, V)$$

be the Grassmannian of two-dimensional subspaces of V. Let

$$0 \longrightarrow \mathscr{S} \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \mathscr{Q} \longrightarrow 0$$

be the tautological sequence, where $\mathscr{S}$ and $\mathscr{Q}$ both have rank 2, and put

$$\mathcal{O}_X(1) = \det \mathscr{Q}.$$

For any sheaf $\mathscr{H}$ on X, write $\mathscr{H}(t) = \mathscr{H} \otimes \mathcal{O}_X(t)$.

Lemma 3. *The canonical bundle of X is*

$$\omega_X \simeq \mathcal{O}_X(-4).$$

Proof. The tangent bundle is

$$\mathscr{T}_X \simeq \mathscr{S}^\vee \otimes \mathscr{Q}.$$

Since $\det \mathscr{S} \otimes \det \mathscr{Q} \simeq \det V \otimes \mathcal{O}_X \simeq \mathcal{O}_X$, the chosen volume form gives

$$\det \mathscr{S}^\vee \simeq \det \mathscr{Q} = \mathcal{O}_X(1).$$

Therefore

$$\det \mathscr{T}_X \simeq (\det \mathscr{S}^\vee)^{\otimes 2} \otimes (\det \mathscr{Q})^{\otimes 2} \simeq \mathcal{O}_X(4).$$

Thus $\omega_X = \det \Omega^1_X \simeq (\det \mathscr{T}_X)^{-1} \simeq \mathcal{O}_X(-4)$. □

Lemma 4. *The Grassmannian $X = \mathrm{Gr}(2, V)$ is Frobenius split.*

Proof. We give an explicit splitting. Under the Plucker embedding

$$X \hookrightarrow \mathbb{P}(\wedge^2 V) \simeq \mathbb{P}^5,$$

choose homogeneous coordinates

$$x_0 = x_{12}, \quad x_1 = x_{13}, \quad x_2 = x_{14}, \quad x_3 = x_{23}, \quad x_4 = x_{24}, \quad x_5 = x_{34}.$$

Then X is the smooth quadric hypersurface cut out by

$$f = x_0x_5 - x_1x_4 + x_2x_3.$$

The formula remains valid in characteristic 2, with the evident interpretation of signs.

Let $R = k[x_0, \ldots, x_5]$. Let

$$T : F_*R \longrightarrow R$$

be the standard Frobenius trace, characterized on monomials by

$$T(F_*x^\alpha) = \begin{cases} x^\beta, & \text{if } \alpha_j = p\beta_j + (p-1) \text{ for every } j, \\ 0, & \text{otherwise.} \end{cases}$$

This is R-linear for the R-module structure on F_*R.

A homogeneous polynomial P of degree $6(p-1)$ defines an $\mathcal{O}_{\mathbb{P}^5}$-linear map

$$\varphi_P : F_*\mathcal{O}_{\mathbb{P}^5} \longrightarrow \mathcal{O}_{\mathbb{P}^5}$$

by sheafifying the p^{-1}-linear map

$$F_*a \longmapsto T(F_*(Pa)).$$

Take

$$P = f^{p-1}(x_1x_2x_3x_4)^{p-1}.$$

The coefficient of $(x_0x_1x_2x_3x_4x_5)^{p-1}$ in P is 1: indeed, the only contributing term is obtained by taking $(x_0x_5)^{p-1}$ from f^{p-1}. Hence

$$\varphi_P(F_*1) = 1,$$

so φ_P is a Frobenius splitting of $\mathbb{P}^5$.

It remains to check that this splitting is compatible with the quadric X. Write

$$h = (x_1x_2x_3x_4)^{p-1}, \quad \text{so that} \quad P = f^{p-1}h.$$

For every homogeneous $a \in R$,

$$\varphi_P(F_*(fa)) = T(F_*(f^p ha)) = f\,T(F_*(ha)) \in (f),$$

where the middle equality uses R-linearity of T and the fact that multiplication by f on F_*R is multiplication by f^p before applying F_*. Thus the ideal sheaf of X is preserved by φ_P, and the splitting descends to X. □

3 The ample bundle

Define

$$\mathscr{E} := F^*\mathscr{Q} \otimes \mathcal{O}_X(1).$$

Then $\mathscr{E}$ has rank 2.

Lemma 5. *The vector bundle $\mathscr{E}$ is ample.*

Proof. Pulling back the tautological quotient

$$V \otimes \mathcal{O}_X \twoheadrightarrow \mathscr{Q}$$

by Frobenius gives a quotient from a trivial rank-four bundle onto $F^*\mathscr{Q}$. After tensoring with $\mathcal{O}_X(1)$, we obtain a quotient

$$\mathcal{O}_X(1)^{\oplus 4} \twoheadrightarrow \mathscr{E}.$$

Since $\mathcal{O}_X(1)$ is ample, the vector bundle $\mathcal{O}_X(1)^{\oplus 4}$ is ample, and every quotient of an ample vector bundle is ample. Hence $\mathscr{E}$ is ample. □

The basic nonvanishing we need is

$$H^2(X, \mathscr{E} \otimes \omega_X) \neq 0.$$

By Lemma 3, this is equivalent to

$$H^2\big(X, F^*\mathscr{Q}(-3)\big) \neq 0.$$

We now prove this nonvanishing.

4 A rank-two Frobenius exact sequence

Lemma 6. *Let $\mathscr{M}$ be a rank-two vector bundle on a scheme of characteristic $p > 0$. There is a functorial short exact sequence*

$$0 \longrightarrow F^*\mathscr{M} \longrightarrow \operatorname{Sym}^p \mathscr{M} \xrightarrow{\delta} \operatorname{Sym}^{p-2} \mathscr{M} \otimes \det \mathscr{M} \longrightarrow 0.$$

Here $F^*\mathscr{M} \to \operatorname{Sym}^p \mathscr{M}$ is the linearization of the p-th power map $m \mapsto m^p$.

Proof. Let

$$d : \operatorname{Sym}^p \mathscr{M} \longrightarrow \operatorname{Sym}^{p-1} \mathscr{M} \otimes \mathscr{M}$$

be the comultiplication, or first-derivative, map. Locally, in a basis x, y, it is given by

$$d(x^a y^{p-a}) = a x^{a-1} y^{p-a} \otimes x + (p-a) x^a y^{p-a-1} \otimes y.$$

The multiplication map

$$\operatorname{Sym}^{p-1} \mathscr{M} \otimes \mathscr{M} \longrightarrow \operatorname{Sym}^p \mathscr{M}$$

composed with d is multiplication by p, hence is zero in characteristic p. Thus d factors through the Koszul kernel

$$\ker\bigl(\operatorname{Sym}^{p-1} \mathscr{M} \otimes \mathscr{M} \to \operatorname{Sym}^p \mathscr{M}\bigr) \simeq \operatorname{Sym}^{p-2} \mathscr{M} \otimes \det \mathscr{M}.$$

Let

$$\delta : \operatorname{Sym}^p \mathscr{M} \longrightarrow \operatorname{Sym}^{p-2} \mathscr{M} \otimes \det \mathscr{M}$$

be the resulting map.

In a local basis x, y, the inclusion of this Koszul kernel is

$$s \otimes (x \wedge y) \longmapsto sy \otimes x - sx \otimes y.$$

Therefore δ is given locally by

$$\delta(x^a y^{p-a}) = \begin{cases} a\, x^{a-1} y^{p-a-1} \otimes (x \wedge y), & 1 \le a \le p-1, \\ 0, & a = 0 \text{ or } a = p. \end{cases}$$

The coefficients a are units for $1 \le a \le p-1$, so δ is locally surjective and its kernel is locally generated by x^p and y^p. This kernel is precisely the image of the canonical map

$$F^*\mathscr{M} \longrightarrow \operatorname{Sym}^p \mathscr{M}, \qquad F^* m \longmapsto m^p.$$

Hence the sequence is exact. □

Applying Lemma 6 to $\mathscr{M} = \mathscr{Q}$ and using $\det \mathscr{Q} = \mathcal{O}_X(1)$ gives

$$0 \longrightarrow F^*\mathscr{Q} \longrightarrow \operatorname{Sym}^p \mathscr{Q} \longrightarrow \operatorname{Sym}^{p-2} \mathscr{Q}(1) \longrightarrow 0.$$

After twisting by $\mathcal{O}_X(-3)$, we obtain

$$0 \longrightarrow F^*\mathscr{Q}(-3) \longrightarrow \operatorname{Sym}^p \mathscr{Q}(-3) \longrightarrow \operatorname{Sym}^{p-2} \mathscr{Q}(-2) \longrightarrow 0. \tag{1}$$

5 The required cohomology calculation

We compute the cohomology of the two outer terms in (1) by passing to the flag variety.

Let

$$Y = \mathrm{Fl}(2, 3; V)$$

be the variety of flags $U \subset H \subset V$ with $\dim U = 2$ and $\dim H = 3$. Let

$$\pi : Y \longrightarrow X = \mathrm{Gr}(2, V), \qquad g : Y \longrightarrow Z := \mathrm{Gr}(3, V)$$

be the two projections. We use the quotient convention for projective bundles.

Then

$$Y \simeq \mathbb{P}_X(\mathscr{Q}),$$

where the relative $\mathcal{O}(1)$ is

$$\mathcal{O}_\pi(1) = V/H.$$

Similarly,

$$Y \simeq \mathbb{P}_Z(\mathscr{T}),$$

where $\mathscr{T} \subset V \otimes \mathcal{O}_Z$ is the tautological rank-three subbundle on Z and

$$\mathcal{O}_g(1) = H/U.$$

Let

$$L := \mathcal{O}_Z(1) = V/H$$

be the quotient-line Plucker bundle on Z. Then on Y one has

$$\mathcal{O}_\pi(1) = g^*L$$

and

$$\pi^*\mathcal{O}_X(1) = g^*L \otimes \mathcal{O}_g(1),$$

since V/U has determinant $(V/H) \otimes (H/U)$.

Lemma 7. *For every $m \geq 0$,*

$$H^\bullet\big(X, \mathrm{Sym}^m \mathscr{Q}(-2)\big) = 0.$$

Moreover, for every $m \geq 0$ and every integer j,

$$H^j\big(X, \mathrm{Sym}^m \mathscr{Q}(-3)\big) \simeq H^{j-2}\big(Z, L^{m-2}\big),$$

with the convention that $H^r = 0$ for $r < 0$.

Proof. Because $Y = \mathbb{P}_X(\mathscr{Q})$ with quotient convention,

$$\pi_*\mathcal{O}_\pi(m) = \mathrm{Sym}^m \mathscr{Q}, \qquad R^a\pi_*\mathcal{O}_\pi(m) = 0 \quad (a > 0, m \geq 0).$$

Thus, for every integer t,

$$\begin{aligned} H^j\big(X, \mathrm{Sym}^m \mathscr{Q}(t)\big) &\simeq H^j\big(Y, \mathcal{O}_\pi(m) \otimes \pi^*\mathcal{O}_X(t)\big) \\ &\simeq H^j\big(Y, g^*L^{m+t} \otimes \mathcal{O}_g(t)\big). \end{aligned}$$

For $t=-2$, the fibers of g are projective planes, and

$$H^a(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)) = 0 \quad \text{for every } a.$$

Therefore

$$R^a g_* \mathcal{O}_g(-2) = 0 \quad \text{for every } a,$$

and it follows that

$$H^\bullet\big(X, \operatorname{Sym}^m \mathscr{Q}(-2)\big) = 0.$$

For $t=-3$, relative duality for

$$g : Y = \mathbb{P}_Z(\mathscr{T}) \longrightarrow Z$$

gives

$$R^a g_* \mathcal{O}_g(-3) = 0 \quad (a \neq 2), \qquad R^2 g_* \mathcal{O}_g(-3) \simeq (\det \mathscr{T})^{-1}.$$

Since $\det \mathscr{T} \otimes L \simeq \det V \otimes \mathcal{O}_Z \simeq \mathcal{O}_Z$, the chosen volume form identifies

$$(\det \mathscr{T})^{-1} \simeq L.$$

Thus

$$R^2 g_*\big(g^* L^{m-3} \otimes \mathcal{O}_g(-3)\big) \simeq L^{m-3} \otimes L \simeq L^{m-2},$$

and all other higher direct images are zero. The Leray spectral sequence therefore gives

$$H^j\big(Y, g^* L^{m-3} \otimes \mathcal{O}_g(-3)\big) \simeq H^{j-2}\big(Z, L^{m-2}\big).$$

Using the first displayed identification with $t=-3$ proves the claim. □

Proposition 8. *One has*

$$H^2\big(X, F^* \mathscr{Q}(-3)\big) \neq 0.$$

Equivalently,

$$H^2(X, \mathscr{E} \otimes \omega_X) \neq 0.$$

Proof. By Lemma 7,

$$H^\bullet\big(X, \operatorname{Sym}^{p-2} \mathscr{Q}(-2)\big) = 0.$$

Therefore the short exact sequence (1) induces isomorphisms

$$H^i\big(X, F^* \mathscr{Q}(-3)\big) \simeq H^i\big(X, \operatorname{Sym}^p \mathscr{Q}(-3)\big)$$

for all i.

Taking $m=p$ and $j=2$ in Lemma 7, we get

$$H^2\big(X, \operatorname{Sym}^p \mathscr{Q}(-3)\big) \simeq H^0\big(Z, L^{p-2}\big).$$

But $Z = \operatorname{Gr}(3, V) \simeq \mathbb{P}^3$ and $L = \mathcal{O}_Z(1)$, so

$$H^0(Z, L^{p-2}) \neq 0$$

for every prime p, including $p=2$. Hence

$$H^2\big(X, F^* \mathscr{Q}(-3)\big) \neq 0.$$

Finally, by Lemma 3,

$$\mathscr{E} \otimes \omega_X = (F^* \mathscr{Q} \otimes \mathcal{O}_X(1)) \otimes \mathcal{O}_X(-4) = F^* \mathscr{Q}(-3),$$

so the equivalent formulation follows. □

6 Proof of the theorem

We now finish the proof of Theorem 1. By Lemma 4, X is Frobenius split. By Proposition 8,

$$H^2(X, \mathscr{E} \otimes \omega_X) \neq 0.$$

Applying Lemma 2 to $Y = X$ and $\mathscr{A} = \mathscr{E}$ gives

$$H^2\big(X, F^{n*}\mathscr{E} \otimes \omega_X\big) \neq 0 \quad \text{for every } n \geq 0.$$

Thus, with

$$\mathscr{G} = \omega_X,$$

we have a fixed coherent sheaf $\mathscr{G}$ such that

$$H^2\big(X, F^{n*}\mathscr{E} \otimes \mathscr{G}\big) \neq 0 \quad \text{for every } n \geq 0.$$

Since $\operatorname{rk}(\mathscr{E}) = 2$, this is nonvanishing in the borderline degree $i = \operatorname{rk}(\mathscr{E})$ for all Frobenius iterates. This proves Theorem 1.

Equivalently, because $F^{n*}\mathcal{O}_X(1) = \mathcal{O}_X(p^n)$,

$$F^{n*}\mathscr{E} \otimes \omega_X \simeq F^{(n+1)*}\mathscr{Q} \otimes \mathcal{O}_X(p^n - 4),$$

and the groups

$$H^2\big(X, F^{(n+1)*}\mathscr{Q}(p^n - 4)\big)$$

are nonzero for every $n \geq 0$.

Remark 9 (Quantitative lower bound). The proof of Lemma 2 shows more than nonvanishing. After restriction of scalars along Frobenius, $H^2(X, \mathscr{E} \otimes \omega_X)$ is a direct summand of $H^2(X, F^{n*}\mathscr{E} \otimes \omega_X)$ for every n. Since the proof of Proposition 8 identifies

$$H^2(X, \mathscr{E} \otimes \omega_X) \simeq H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(p-2)),$$

one gets

$$\dim_k H^2\big(X, F^{n*}\mathscr{E} \otimes \omega_X\big) \geq \binom{p+1}{3}$$

for all $n \geq 0$, up to the harmless Frobenius twist of the k-vector-space structure.

Remark 10 (Why symmetric powers do not force vanishing). There is always an inclusion

$$F^{n*}\mathscr{E} \hookrightarrow \operatorname{Sym}^{p^n} \mathscr{E}$$

locally given by $e \mapsto e^{p^n}$. For an ample vector bundle $\mathscr{E}$, Serre vanishing on $\mathbb{P}_X(\mathscr{E})$ implies that, for a fixed coherent sheaf $\mathscr{H}$,

$$H^j\big(X, \operatorname{Sym}^m \mathscr{E} \otimes \mathscr{H}\big) = 0 \quad (j > 0,\ m \gg 0).$$

The example above shows that this does not imply vanishing for the Frobenius subbundle $F^{n*}\mathscr{E}$. In the present example,

$$H^2\big(X, \operatorname{Sym}^{p^n} \mathscr{E} \otimes \omega_X\big) = 0 \quad \text{for } n \gg 0,$$

while

$$H^2\big(X, F^{n*}\mathscr{E} \otimes \omega_X\big) \neq 0 \quad \text{for every } n \geq 0.$$

The quotient $\operatorname{Sym}^{p^n} \mathscr{E}/F^{n*}\mathscr{E}$ therefore carries essential cohomological information.