Introduction to O-minimality

UGA
October 9, 2020

Pila–Bombieri

Theorem (Pila–Bombieri)

Let $C \subset \mathbb{R}^2$ be a compact real analytic curve which is not algebraic. Then for any $\epsilon > 0$, there is a constant $a(C, \epsilon)$ such that

$$|tC \cap \mathbb{Z}^2| \leq a(C, \epsilon)t^{\epsilon} \text{ for } t \geq 1$$

Idea:

- For any real analytic function f on $[0, N]$, the integer points on the graph lie on a union of $(||f||^{1/2}N)^{\frac{8}{3(d+3)}}$ degree d algebraic curves, where $||f||$ is the normalized maximum of all order derivatives.
- The space of degree d curves is compact.

FALSE if C is not compact: take graph of $\sin(2\pi x)$. Need **some** tameness property of C.

Structures

Definition

A structure S is a collection $(S_n)_{n \in \mathbb{N}}$ where S_n is a set of subsets of $\mathbb{R}^n$ satisfying the following conditions:

- Each S_n is closed under finite intersections, unions, and complements;
- The collection (S_n) is closed under finite Cartesian products and coordinate projection;
- For every polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$, the zero set

$$(P = 0) := \{x \in \mathbb{R}^n \mid P(x) = 0\} \subset \mathbb{R}^n$$

  is an element of S_n.

We say:

- $U \subset \mathbb{R}^n$ with $U \in S_n$ is a (S-)definable subset of $\mathbb{R}^n$;
- for $U \in S_n$, and $V \in S_m$, $f : U \to V$ is (S-)definable if the graph is.

First (non)examples

Non-Example

The collection S of real algebraic sets—that is, $S_n =$ the Boolean algebra generated by sets of the form $(P = 0)$ for $P \in \mathbb{R}[x_1, \ldots, x_n]$—is *not* a structure. Indeed, for any $P \in \mathbb{R}[x_1, \ldots, x_n]$, the image of the projection of $(x_0^2 = P)$ forgetting x_0 is $(P \geq 0)$.

Example

Let $\mathbb{R}_{\mathrm{alg}}$ be the collection of real semi-algebraic subsets of $\mathbb{R}^n$—that is, $(\mathbb{R}_{\mathrm{alg}})_n$ is the Boolean algebra generated by sets of the form $(P \geq 0)$ for $P \in \mathbb{R}[x_1, \ldots, x_n]$. Then $\mathbb{R}_{\mathrm{alg}}$ *is* a structure. By the Tarski–Seidenberg theorem, coordinate projections of real semi-algebraic sets are real semi-algebraic, and the other axioms are easy to verify. $\mathbb{R}_{\mathrm{alg}}$ is therefore a structure, in fact the structure generated by real algebraic sets.

First order logic

Remark

Tarski–Seidenberg is usually phrased as quantifier elimination for the real ordered field.

The axioms for a structure say definable sets are closed under first order formulas, as intersections, unions, and complements correspond to the logical operators "and", "or", and "not", while the projection axiom corresponds to universal and existential quantifiers.

Moreover, we can make the same definition for any real closed field, and base-change to these fields plays a similar role to base-changing to generic points of schemes in algebraic geometry. We won't say much about it, but it is a useful perspective to keep in mind.

Euclidean topology

Proposition

Let S be a structure, and endow $\mathbb{R}^n$ with the euclidean topology. Closures, interiors, and boundaries of definable sets are definable.

Proof.

We just show that the closure of a definable set $U \subset \mathbb{R}^n$ is defined by a first order formula and leave the rest as an exercise:

$$\overline{U} = \left\{ x \in \mathbb{R}^n \;\middle|\; \forall \epsilon > 0, \exists y \in U \text{ s.t. } \sum_i (x_i - y_i)^2 < \epsilon \right\}$$

□

O-minimality

Definition

A structure S is said to be o-minimal if $S_1 = (\mathbb{R}_{\mathrm{alg}})_1$—that is, if the S-definable subsets of the real line are exactly finite unions of intervals.

Example

$\mathbb{R}_{\mathrm{alg}}$ is o-minimal, clearly.

Example

Let $\mathbb{R}_{\mathsf{sin}}$ be the structure generated by the graph of $\mathsf{sin} : \mathbb{R} \to \mathbb{R}$. $\mathbb{R}_{\mathsf{sin}}$ is not o-minimal as $\pi\mathbb{Z} = \mathsf{sin}^{-1}(0)$ is definable and infinite.

Example

Let $\mathbb{R}_{\mathsf{exp}}$ be the structure generated by the graph of the real exponential $\mathsf{exp} : \mathbb{R} \to \mathbb{R}$. $\mathbb{R}_{\mathsf{exp}}$ is o-minimal by a result of Wilkie. Quantifier elimination does not hold for $\mathbb{R}_{\mathsf{exp}}$.

Definable sets can be complicated!

Example

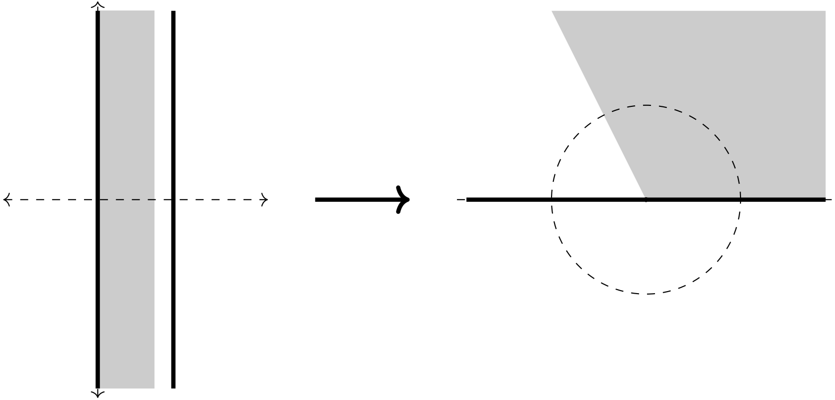
Let $\mathbb{R}_{\mathrm{an}}$ be the structure generated by the graphs of all restrictions $f|_{B(R)}$ of real analytic functions $f : B(R') \to \mathbb{R}$ on a finite radius $R' < \infty$ open euclidean ball (centered at the origin) to a strictly smaller radius $R < R'$ ball. $\mathbb{R}_{\mathrm{an}}$ is o-minimal by van-den-Dries, using Gabrielov's theorem of the complement. Note that while $\sin(x)$ is not $\mathbb{R}_{\mathrm{an}}$-definable, its restriction to any finite interval is. Via the embedding $\mathbb{R}^n \subset \mathbb{R}\mathrm{P}^n$, this is equivalent to the structure of subsets of $\mathbb{R}^n$ that are subanalytic in $\mathbb{R}\mathrm{P}^n$.

Example

Let $\mathbb{R}_{\mathrm{an,exp}}$ be the structure generated by $\mathbb{R}_{\mathrm{an}}$ and $\mathbb{R}_{\exp}$. $\mathbb{R}_{\mathrm{an,exp}}$ is o-minimal by a result of van-den-Dries–Miller. Most of the applications to algebraic geometry currently use the structure $\mathbb{R}_{\mathrm{an,exp}}$.

Definable fundamental sets I

Consider the exponential as the quotient map $e^{2\pi i z} : \mathbb{C} \to \mathbb{C}^* = \mathbb{Z}\backslash\mathbb{C}$.



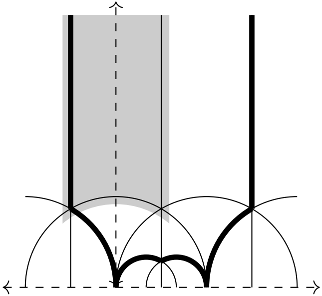
$$e^{2\pi i z} = e^{-2\pi y}\cos(2\pi x) + ie^{-2\pi y}\sin(2\pi x)$$

$$e^{2\pi i z} : F \to \mathbb{C}^* \text{ is } \mathbb{R}_{\mathrm{an,exp}}\text{-definable!}$$

Definable fundamental sets II

$$\Gamma(2) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv 1 \mod 2\}$$

$\pi : \mathbb{H} \to Y(2) = \Gamma(2)\backslash\mathbb{H}$ the quotient.



$Y(2) \subset \mathbb{C}^N$ is algebraic

$\pi : F \to Y(2)$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable!

Definable sets have nice structure

Definition

A *definable cell decomposition* of $\mathbb{R}^n$ is a partition $\mathbb{R}^n = \bigsqcup D_i$ into finitely many pairwise disjoint definable subsets D_i, called cells, of the form:

1. $\underline{n = 0}$: There is exactly one definable cell decomposition of $\mathbb{R}^0$. Its unique cell is all of $\mathbb{R}^0$.
2. $\underline{n > 0}$: Write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. There is a cell decomposition $\{E\}$ of $\mathbb{R}^{n-1}$ and for each E we have: an integer $m_E \in \mathbb{N}$ and continuous definable functions $f_{E,k} : E \to \mathbb{R}$ for each $0 < k < m_E$ such that

$$f_{E,0} := -\infty < f_{E,1} < \cdots < f_{E,m_E-1} < f_{E,m_E} := +\infty.$$

   The cells are:

- graphs: $\{(x, f_{E,k}(x)) \mid x \in E\}$ for each E and $0 < k < m_E$;
- bands: $(f_{E,k}, f_{E,k+1}) := \{(x, y) \mid x \in E \text{ and } y \in (f_{E,k}(x), f_{E,k+1}(x))\}$ for each E and $0 \leq k < m_E$.

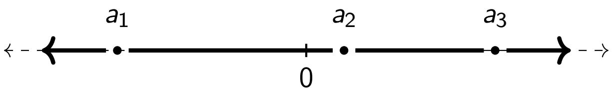
A cell decomposition of $\mathbb{R}$

In this case, there is $m \in \mathbb{N}$ and $a_k \in \mathbb{R}$ for each $0 < k < m$ such that

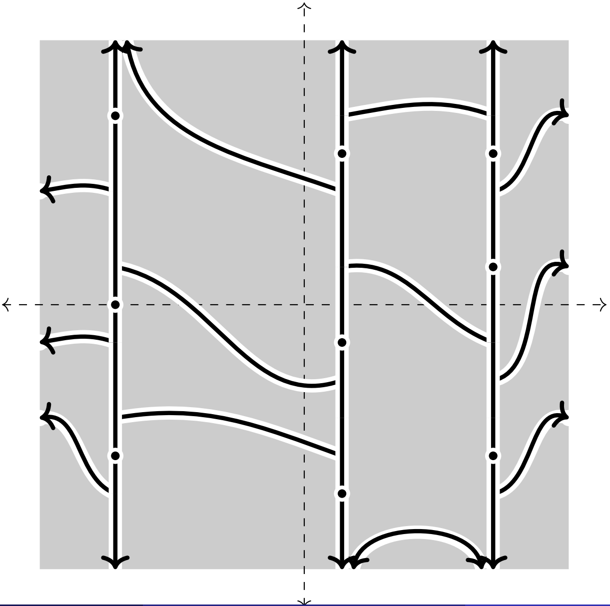
$$a_0 := -\infty < a_1 < \cdots a_{m-1} < a_m := +\infty$$

and the cells are:

- $\{a_k\}$ for $0 < k < m$;
- (a_k, a_{k+1}) for $0 \leq k < m$.



A cell decomposition of $\mathbb{R}^2$



Cell decomposition

Theorem

For any finite collection $U_j \subset \mathbb{R}^n$ of definable sets, there is a definable cell decomposition of $\mathbb{R}^n$ such that each U_j is a union of cells.

Corollary/Definition

For any definable set $U \subset \mathbb{R}^n$ we define $\dim U$ to be the largest dimension of its cells with respect to a definable cell decomposition.

Uniform boundedness

Corollary

Let $f : U \to V$ *be a definable map. Then for each* $n \in \mathbb{N}$*, the subset* $V_n := \{v \in V \mid \dim f^{-1}(v) = n\} \subset V$ *is definable.*

Proof.

Consider the graph, and order the coordinates *backwards*. As is clear from the inductive definition, each cell has constant dimension over its projection. □

Corollary

Let $f : U \to V$ *be a definable map with finite fibers. Then for each* $n \in \mathbb{N}$*, the subset* $V_n := \{v \in V \mid \# f^{-1}(v) = n\} \subset V$ *is definable. Moreover, the size of the fibers is uniformly bounded.*

Algebraization theorems

Definability $\Rightarrow$ algebraicity in two important ways:

- **Arithmetic:** Definable sets with many rational points are algebraic (Pila–Wilkie);
- **Complex-analytic:** Definable complex analytic subvarieties are algebraic (Peterzil–Starchenko).

Heights

Definition

The (archimedean) *height* $H(r)$ of a rational number $r \in \mathbb{Q}$ is defined to be $\max(|a|, |b|)$, where $r = a/b$ for coprime integers a, b. Likewise, for $\alpha \in \mathbb{Q}^n$ we define the height to be $H(\alpha) = \max H(\alpha_i)$.

For $U \subset \mathbb{R}^n$ we define:

$$\begin{aligned} N(U,t) &:= \#\{\alpha \in U \cap \mathbb{Q}^n \mid H(\alpha) \le t\} \\ U^{\mathrm{alg}} &:= \bigcup_{\substack{Z \text{ connected semi-algebraic} \\ \dim Z > 0 \\ Z \subset U}} Z \\ U^{\mathsf{tr}} &:= U \setminus U^{\mathrm{alg}}. \end{aligned}$$

Note that U^{alg} may well *not* be definable in any o-minimal structure even if U is.

The counting theorem

Theorem (Pila–Wilkie)

Let $U \subset \mathbb{R}^n$ be definable in an o-minimal structure. Then for any $\epsilon > 0$,

$$N(U^{\mathrm{tr}}, t) = O(t^{\epsilon}).$$

The counting theorem is often used to deduce from the presence of many rational points on U the existence of a *semialgebraic* subset $Z \subset U$ with many rational points.

Idea: echo the proof from before using the uniformity coming from definability in place of compactness.

Definable Chow

Theorem (Peterzil–Starchenko)

Let $Z \subset \mathbb{C}^N$ be a closed analytic subvariety whose underlying set is definable in an o-minimal structure. Then Z is algebraic.

Baby case: any definable entire function $f : \mathbb{C} \to \mathbb{C}$ is algebraic by Casorati–Weierstrass!

Thanks!