Galois actions on Tate modules + cohomology

Last time: Padma used:

<u>Thm</u> (Serre) A is an AV/K - f.g. field of char 0

$$\rho: \mathrm{Gal}(\bar{K}/K) \to GL(T(A))$$

scalar matrizes

$\varprojlim_n A[n]$

$\cong \hat{\mathbb{Z}}^{2g}$

Then $\hat{\mathbb{Z}}^\times / \mathrm{im}\,\rho \cap \hat{\mathbb{Z}}^\times$ has finite exponent.

Slogan: "many" Galois homotheties.

Today: Context for this thm
+ discussion about Galois actions on Tate modules and cohomology.

(1) Galois rep'ns

X - variety$/K$ - # field

$\rightsquigarrow H^i(X_{\bar{K}, \acute{e}t}, \mathbb{Q}_p)$ ← p-adic étale cohomology

cts

finite dim'l v.s. w/ Gal($\bar{K}/K$)-action.

Ex A - AV $H^1(A, \mathbb{Q}_p) = T_p(A)^\vee \otimes \mathbb{Q}_p$.

Properties of these rep'ns:

(1) Unramified at almost all places v of K.

$$\begin{tikzcd} I_v \arrow[r, hook] \arrow[rrr, bend right, "\text{trivial} \iff \text{unramified at } v."'] & \mathrm{Gal}(\bar{K}_v/K_v) \arrow[r, hook] & \mathrm{Gal}(\bar{K}/K) \arrow[r] & GL(V) \end{tikzcd}$$

Ex X sm. proper variety, good reduction at v

Then $H^i(X_{\bar{K}}, \mathbb{Q}_p)$ is unramified at v if v is not a place above p.

(2) For v above p (p-adic Hodge theory)

$$\rho|_{\mathrm{Gal}(\bar{K}_v/K_v)} : \mathrm{Gal}(\bar{K}_v/K_v) \to GL(V)$$

Hodge-Tate < is "de Rham" $\Rightarrow$ "potentially semistable"

X sm. proper w/ good reduction $\Rightarrow$ "crystalline"

(i) Morally (X sm. proper w/ good reduction)

this means $\rho|_{\mathrm{Gal}(\bar{K}_v/K_v)}$ can be computed

from de Rham coh. of X_{K_v} +
crystalline coh. of $X \otimes k(v)$.

(ii) Hodge-Tate means

$$V \otimes_{\mathbb{Q}_p} \mathbb{C}_p = \oplus \, \mathbb{C}_p(-j)^{\oplus n_j}$$

$\curvearrowleft \widehat{\overline{\mathbb{Q}}_p}$

$Gal(\bar{K_v}/K_v) \subseteq Gal(\bar{K_v}/\mathbb{Q}_p)$

$\mathbb{C}_p$-semilinear Gal-reps

$$\mathbb{Z}_p(i) = \chi_{cyc}^{\otimes i}, \quad \mathbb{C}_p(i) = \mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$$

<u>Rem</u> X sm. proper : $n_j = \dim H^{i-j}(X, \Omega^j)$

$$V = H^i(X_{\bar{K}}, \mathbb{Q}_p)$$

$$\Omega^j = \bigwedge^j \Omega^1_X$$

<u>Defn</u> If $Gal(\bar{K}/K)$ satisfies (1)+(2)
we say it's "geometric in the sense of Fontaine-Mazur."

<u>Conj</u> Geometric rep'ns come from geometry.

(3) <u>Thm</u> (Bogomolov) $\rightarrow GL_n(\mathbb{Z}_p)$

$\rho: \mathrm{Gal}(\bar{k}/k) \to GL_n(\mathbb{Q}_p)$ cts rep'n

which is geometric. $\mathrm{im}\,\rho$ is open in the $\mathbb{Q}_p$-pts of its Zariski-closure.

$(\mathrm{Lie}\ \mathrm{im}\,\rho = \mathrm{Lie}\ \overline{\mathrm{im}\,\rho}.)$

<u>Non-ex</u>:

$$\begin{tikzcd} \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \arrow[r, "\chi_{cyc}"] \arrow[ddrrr, bend right] & \hat{\mathbb{Z}}^\times \arrow[r] & \mathbb{Z}_p^\times \arrow[r] & \mathbb{Z}_p \arrow[d] \\ & & & \mathbb{Z}_p^2 \arrow[d] \\ & & & \mathbb{G}_m^2(\mathbb{Q}_p) \end{tikzcd}$$

(4) $\rho: \mathrm{Gal}(\bar{k}/k) \to GL(H^i(X_{\bar{k},\text{ét}}, \mathbb{Q}_p))$

Then ρ is "mixed": $\exists!$ increasing filtration $W_\bullet$ on H^i s.t. almost all Frobenii act on $\mathrm{gr}^W_j H^i$ w/ eigenvalues alg. #'s of absolute value $q^{-j/2}$ ← # res. field.

Deligne: Weil I + Théorie de Hodge II + III).

X sm. proper: H^i is <u>pure</u> of wt i.

$\mathrm{gr}^W_i H^i = H^i$.

2) Homotheties

<u>Thm</u> (Bogomolov) X sm. proper variety$/K$-# field

$$\rho: G_K \to GL(H^i(X_{\bar{K}}, \mathbb{Q}_\ell))$$

Then if $i \geq 1$, $[\mathrm{im}\,\rho \cap \mathbb{Z}_\ell^\times : \mathbb{Z}_\ell^\times] < \infty$.

<u>Slogan</u>: lots of ℓ-adic homotheties.

<u>Pf</u> (i) ETS $\mathrm{im}\,\rho$ contains $\mathbb{Q}_\ell^\times$

scalar matrices

by Bogomolov's open image Thm

(ii) Choose v of K not over ℓ

where X has good reduction.

Let $F = F_v$ be Frobenius at v.

(iii) Can assume all eigenvalues of are

alg. #'s w/ abs. value $q^{-i/2}$.

(Simplifying assumption: F is diagonal)

(iv) $T = \overline{\{F^n\}_{n \in \mathbb{Z}}} \hookleftarrow T^0$ is

a torus.

(v) $X_F \subseteq X^*(T)$

$$= \{\chi \mid \chi(F) = 1\}$$

$$T = \{M \in \text{Diag} \mid \chi(M) = 1 \text{ for all } M \in X_F\}$$

(vi) Want: diagonal matrices in this sp

$$\chi \in X_F$$

$$\chi(\lambda_1, \dots, \lambda_n) = \lambda_1^{a_1} \cdots \lambda_n^{a_n} = 1$$

← eigenvalues of F

$\Downarrow$

$$\chi(\zeta^{i\nu_1}, \dots, \zeta^{i\nu_2}) \cdot \chi(|\lambda_1|, \dots, |\lambda_n|) = |\lambda_1|^{a_1} \cdots |\lambda_n|^{a_n} = 1$$

← scalar. $\square$