UNLIKELY INTERSECTIONS IN MULTIPLICATIVE GROUPS

(& ZILBER'S CONJECTURE)

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Recall from last time

Setup for "unlikely intersections":

ambient space (ex: semi-ab. var.)

special subset

(ex: Zariski closure of some set of pts.)

class of subspaces

(subgps./cosets)

what can we say about the intersections?

Today's ambient space = algebraic tori.

In AV's case, natural motivation was, eg: Manin - Mumford conj. (Raynaud's thm.)

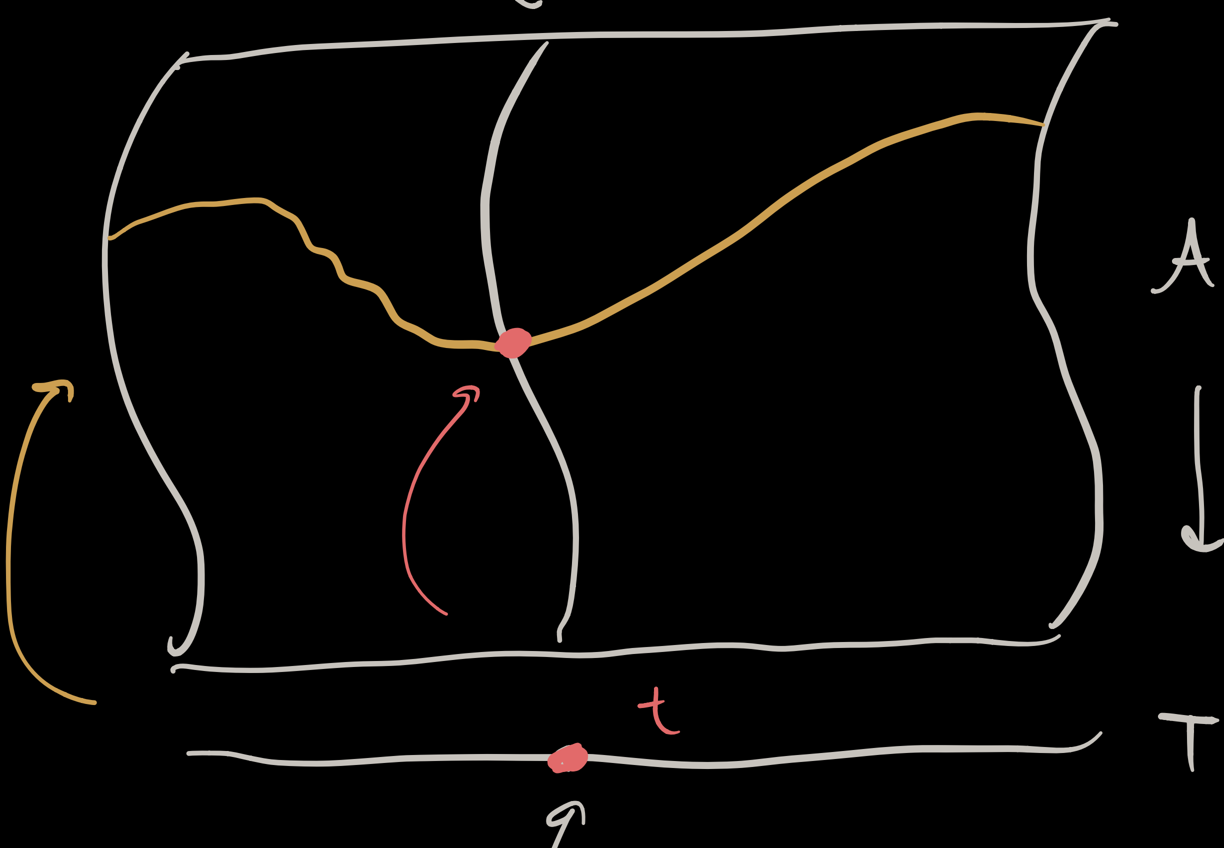
What about tori?

I) Motivation:

Let $X/\overline{\mathbb{Q}}$ be a curve.

$A \to T$ a family of abelian varieties.

("specialization".



($\bar{\mathbb{Q}}$-point

Have a "specialization map":

$$\sigma_t : A(T) \longrightarrow A_t(\bar{\mathbb{Q}})$$

<u>Thm. (Silverman)</u>:

$\{ t \in T(\bar{\mathbb{Q}}) \mid \sigma_t$ is not injective $\}$ is a set of bounded ht.

(If X not a curve, Shioda/: set of $t \in T(\bar{\mathbb{Q}})$

Serre for which σ_t

not injective is thin.

A similar theorem:

Thm. (Manin - Demjanenko):

(See Serre, Pg. 62).

K a number field.

X/K nice variety.

A/K an A.V.

$f_1, \dots, f_m \in \mathrm{Mor}(X, A)$.

Assume:

(a) f_i are linearly independent in $A_X(X)$

X-valued pts of A!

(b) rank $NS(X) = 1$

(eg: X a curve).

Then: $\{ t \in X(K) \mid \sigma_t$ not inj $\}$

is FINITE!

Question:

If we replace $A \to X$

by $\mathbb{G}_{m,X} \longrightarrow X$,

what can we say

about

$\mathcal{E} = \{ t \in X(\overline{\mathbb{Q}}) \mid \sigma_t$

not injective $\}$?

<u>Recall</u> :

$\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$.

For any X,

- $\operatorname{Mor}(X, \mathbb{G}_m) = \mathcal{O}_X(X)^\times$

send to a unit in $\mathcal{O}_X(X)$.

- $\operatorname{Mor}(X, \mathbb{G}_m^n)$

$= \operatorname{Mor}(X, \mathbb{G}_m)^n$

$= \left(\mathcal{O}_X(X)^\times\right)^n$

Ex: $X = \mathbb{G}_m^n$, then

any $\varphi : \mathbb{G}_m^n \to \mathbb{G}_m$

looks like:

$$(x_1, \dots, x_n) \mapsto \underset{\in K^\times}{c}\, x_1^{a_1} \cdots x_n^{a_n}$$

$$\overset{!!}{\leadsto} c\, x^a$$

i.e: $\mathrm{Mor}(\mathbb{G}_m^n, \mathbb{G}_m) = K^\times \cdot x_1^{\mathbb{Z}} \cdots x_n^{\mathbb{Z}}$

$\vee\!/$

$\mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m) = x_1^{\mathbb{Z}} \cdots x_n^{\mathbb{Z}}$.

$\cong \mathbb{Z}^n$.

Note: If

$\varphi : (x_1, \dots, x_n) \mapsto x^a$,

then

$\ker \varphi = \{x^a = 1\} \subseteq \mathbb{G}_m^n$

cosets:

$\varphi^{-1}(c) = \{x^a = c\}$

is called a torsion coset

if $c \in \mathbb{G}_m(\overline{\mathbb{Q}})_{\text{tors}}$,

i.e.: c is a root of unity.

Ex: Any $\varphi : \mathbb{G}_m^n \to \mathbb{G}_m^r$

looks like

$$(x_1, \dots, x_n) \mapsto \begin{pmatrix} x^{a_1} \\ \vdots \\ x^{a_r} \end{pmatrix}$$

$$\ker \varphi : \{x^{a_1} = \dots = x^{a_r} = 1\}$$

coset:

$$\varphi^{-1}(c_1, \dots, c_r) : \left\{ \begin{array}{l} x^{a_1} = c_1 \\ \vdots \\ x^{a_r} = c_r. \end{array} \right\}$$

FACT: All subgps. of $\mathbb{G}_m^n$ arise in this way.

Moreover,

$$\operatorname{codim}(\ker \varphi) = \operatorname{rk}_{\mathbb{Z}} \langle a_1, \dots, a_r \rangle$$

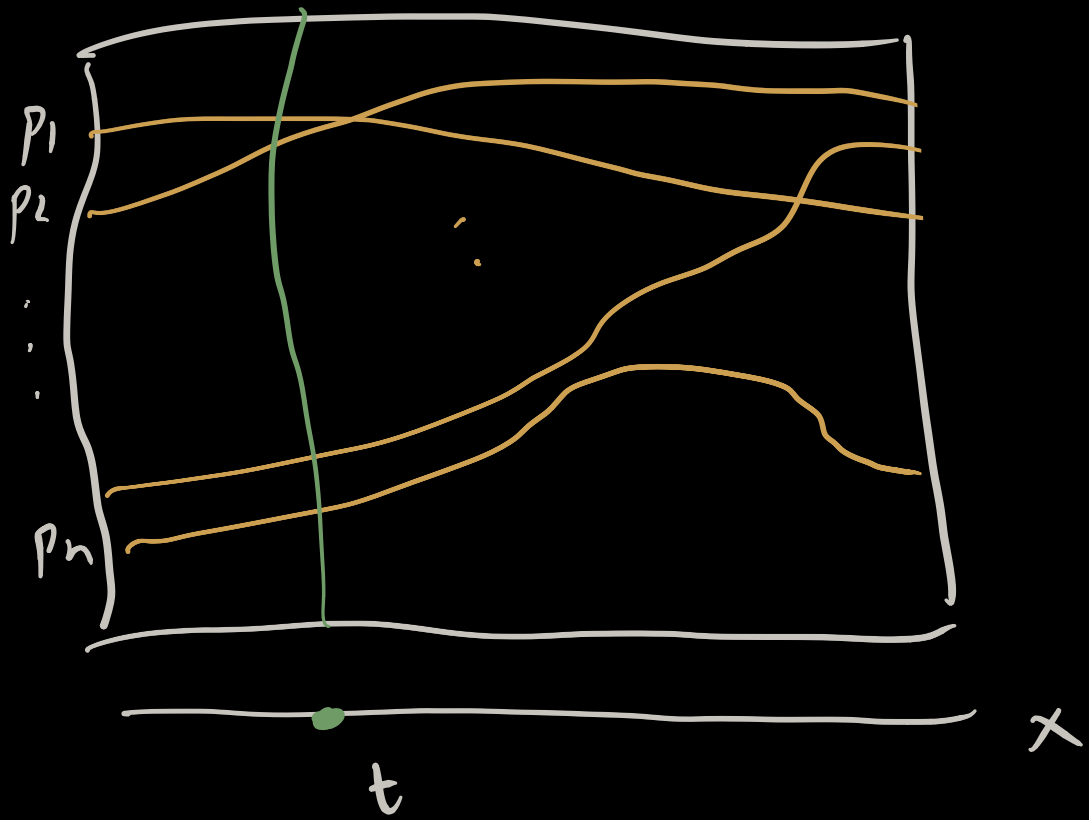
$$\langle a_1, \dots, a_r \rangle \subseteq \mathbb{Z}^r$$

\# of "independent multiplicative relations" satisfied by the coordinates of any point of $\ker \varphi$.

Setup: X an irred. var.

$p_1, \dots, p_n : X \to \mathbb{G}_m$

i.e. $X \to \mathbb{G}_{m,X}$.



Let $X_{(d)}$:=

image of $p_1, \cdots, p_n$

$\{ t \in X(\bar{\mathbb{Q}}) \mid \sigma_t : \mathbb{G}_m(X) \to$

$\mathbb{G}_m(\bar{\mathbb{Q}})$

is of rank $\leq d \}$.

$:= \{ t \in X(\bar{\mathbb{Q}}) \mid p_1(t), \cdots, p_n(t)$

$\in \mathbb{G}_m(\bar{\mathbb{Q}}) = \bar{\mathbb{Q}}^\times$

satisfy $\geq n-d$ indep. relations }

if $X \xrightarrow[\text{closed}]{(p_1,\dots,p_n)} \mathbb{G}_m^n$, then we can identify X w/ its image in $\mathbb{G}_m^n$:

$$= \{(x_1, \dots, x_n) \in X \subseteq \mathbb{G}_m^n \mid x_i\text{'s generate an alg. subgp. of rank } d\}$$

rank d $=$ codim. $n-d$.

$$= \bigcup_{\substack{\dim H \leq d \\ H \text{ alg. subgp. of } \mathbb{G}_m^n}} X \cap H \quad \big|$$

Thus, we have:

$$X_{(0)} \subset X_{(1)} \subset \cdots \subset X_{(n-1)} \subset X_{(n)} = X.$$

Specialization	Zannier-style
$p_i : \operatorname{Spec} K(X) \to \mathbb{G}_{m,K(X)}$ satisfy $p_1^{a_1} \cdots p_n^{a_n} = c \in K^\times \leq K(X)^\times = \mathbb{G}_m(K(X))$	Image of $X \hookrightarrow \mathbb{G}_m^n$ is contained in the proper coset $\{x^a = c\}$

Ex: $X \subseteq \mathbb{A}^1_t$

$$\iota : X \hookrightarrow \mathbb{G}_m^2$$

$$t \longmapsto (t, 1-t).$$

$$X_{(1)} = \{ p \in X(\overline{\mathbb{Q}}) \mid \iota(p)^{a_1} \cdot \iota(p)^{a_2} = 1, \ (a_1, a_2) \neq (1,1) \}.$$

Theorem 1.2.

(Bombieri - Masser - Zannier, 99)

$X \subset \mathbb{G}_m^n$ irred., curve, not contained in a proper coset.

Then the (logarithmic) Weil height of $X_{(n-1)}(\overline{\mathbb{Q}})$ is bdd.!

Assume X/K, K a # fld.

Cor: If $p_1, \dots, p_n \in \mathbb{G}_m(X)$ are linearly indep. modulo constants.

Then

$\{ t \in X(K) \mid p_1(t), \dots, p_n(t)$ dependent in $K^\times \}$

is $\underline{\text{FINITE}}$.

Before proof (sketch), recall:

If $\alpha \in k \subset \overline{\mathbb{Q}}$, then

$$H(\alpha) := \prod_{v \in M_k} \sup(1, |\alpha|_v)$$

(Can normalize valuations so that indep. of k).

Def: Logar. Weil ht. of α is:

$$h(\alpha) := \log H(\alpha).$$

Properties:

- $h(x^{-1}) = h(x)$
- $h(xy) \leq h(x) + h(y)$
- $h(x+y) \leq h(x) + h(y) + \log 2$

- $h(x) = 0$ IFF x is root of unity!!

For $z = (x_1, \dots, x_n) \in \mathbb{G}_m(\overline{\mathbb{Q}})$,

$$h(z) := h(x_1) + \cdots + h(x_n).$$

- Defines an actual norm ($h(x-y)$) on $(\overline{\mathbb{Q}}^\times)^n/\text{tors}$.

- Northcott Property:

$\exists$ only finitely many $\alpha \in \overline{\mathbb{Q}}$ of bdd. ht. & bdd. degree ($/\mathbb{Q}$).

Pf. sketch of Thm. 1.2

Pick $z = (z_1, \dots, z_n) \in X_{(n-1)}$

$\widehat{G_m^n}(\overline{\mathbb{Q}})$

$$\boxed{\text{Goal: Bound } h(z) = h(z_1) + \cdots + h(z_n)}$$

- Write

$$z_1^{m_1} \cdots z_n^{m_n} = 1 \quad \text{— } (*)$$

Let B be a large integer (to be chosen later).

- Can find a the integer $q \leq B^{n-1}$ s.t. that

$$\left| q \frac{m_i}{m_n} - b_i \right| \leq B^{-1},$$

for some $b_i \in \mathbb{Z}$.

Assume WLOG that $m_n = \max |m_i|$

Set $\delta_i := b_i m_n - m_i q$

so $|\delta_i| \leq \underline{\tilde{B}\, m_n}$

no dependence on $m_1, \dots, m_{n-1}$!

Raising $(*)$ to the power q:

$$\prod_i x_i^{\cancel{m_i q}\ 1} = \prod_i x_i^{(b_i \cdot m_n - \delta_i)}$$

$$\Rightarrow \prod_i \left(x_i^{b_i}\right)^{m_n} = \prod_i x_i^{\delta_i}$$

<u>To me</u>: write z_i instead of x_i.

& apply h:

$$m_n\, h\left(\prod_i \left(x_i^{b_i}\right)\right) = \sum_i \delta_i \cdot h(x_i)$$

$$\Rightarrow h\left(\prod_i \left(x_i^{b_i}\right)\right) = \sum_i \frac{\delta_i}{m_n} \cdot h(x_i)$$

$$\leq \tilde{B} \cdot \sum_i h(z_i)$$

$= B^{-1} \cdot h(z)!$

(so far: bdd. ht. of some "approx$^{\underline{n}}$" of original relation by ht. of z).

Also: $|b_i| \leq \left|2\frac{m_i}{m_n}\right| + |B^{-1}|$

$\leq B^{n-1} + 1$,

so $\psi := \prod x_i^{b_i}$ is a non-constant $f^{\underline{n}}$ on X.

- Consider irreducible algebraic relations

$$F_j(\psi, x_j) = 0,$$

s.t. $\deg_{\psi} \leq \deg x_j$

(as $f^{\underline{n}}$ on X)

props of ht.

$\Rightarrow \quad h(z_j) \le c_j \cdot h(\psi(z)) + O_B(1)$

$\le c_j B^{-1} h(z) + O_B(1)$

$\Rightarrow \quad h(z) \le B^{-1} \cdot \left(\sum_j c_j\right) \cdot h(z) + O_B(1)$

$$\Rightarrow \quad h(z) \underbrace{\left(1 - B^{-1} \cdot \sum_j c_j\right)}_{} = O_B(1)$$

Choose B so that this is < 1.

$\blacksquare$

Q: Why assume X not contained in cosets?

Ex: Assume $x_1^{a_1} \cdots x_n^{a_n} = c \neq 0$ on $X \subset \mathbb{G}_m^n$.

Assume $h(c) > 0$

(i.e: $X \not\subseteq$ torsion coset)

Set $H_b := \{ x_1 = (x_1^{a_1} \cdots x_n^{a_n})^b \}$

for $b \in \mathbb{N}$.

Then $H_b \cap X$

$$= \{ z \in X \mid z_1 = c^b \}$$

$$h(z) \geq h(z_1) = b \cdot h(c)$$

$$\to \infty \text{ as } b \to \infty !$$

Rem: $X_{(n-1)}$ always infinite.

Next: X still a curve,

but $X_{(n-2)}$: truly unlikely!

1.3, BMZ 99.

Theorem: $X / \overline{\mathbb{Q}}$ an irred.

curve, not contained in

any proper coset, then

$$X_{(n-2)} \text{ is } \underline{\text{FINITE}}.$$

<u>Sample application</u>:

$\exists$ only finitely many $t \in \mathbb{Q}$ s.that $t, t+1, t-1$ satisfy two indep. multiplicative relations

$$(t-1)^a t^b (t+1)^c = 1$$

<u>Ex</u>: Consider

$$X \hookrightarrow \mathbb{G}_m^5$$

$$t \mapsto (t-1, t, t+1, 2, 3)$$

Image is contained in the proper (non-torsion!) coset $(1,1,1,2,3) \cdot H$, where $H : \{x_4 = x_5 = 1\}$.

Is $X_{(5-2)} = X_{(3)}$ finite?

Yes, b/c:

Thm (Maurin, '08)

$X/\overline{\mathbb{Q}}$ irred. curve, not contained in any proper subgp. $\subseteq \mathbb{G}_m^n$, then $X_{(n-2)}$ is FINITE.

Ex: Take $\mathbb{G}_m^m(\overline{\mathbb{Q}})$

$$X = Y \times (\alpha_1, \dots, \alpha_m) \subseteq \mathbb{G}_m^n$$

Y curve in $\mathbb{G}_m^{n-m}$, $\alpha_i \in \mathbb{G}_m^m(\overline{\mathbb{Q}})$

Let $H \leq \mathbb{G}_m^n$

$$x_1^{-a} \cdot x_{n-m+1}^{a_1} \cdots x_n^{a_m}$$

$$= x_2^{-b} \cdot x_{n-m+1}^{b_1} \cdots x_n^{b_m} = 1$$

what is $X \cap G$?

$$\{(y_1, \dots, y_{n-m}) \in Y(\overline{\mathbb{Q}}) \mid \left\{\begin{array}{l} y_1 = \alpha_{n-m+1}^{a_1/a} \cdots \alpha_n^{a_m/a} \\ y_2 = \alpha_{n-m+1}^{b_1/b} \cdots \alpha_n^{b_m/b} \end{array}\right\}$$

in division subgp. of $\mathbb{G}_m^m$ generated by α_i's!!

(Maurin) $\Rightarrow$ if $(y_1, y_2) \in \mathbb{G}_m^2$ is not inside a proper subgp.,

then

Finally, consider X of higher dimension.

Natural generalization of Thm. 1.2 to higher dim. X:

<u>Conj</u>: Let $X \subset \mathbb{G}_m^n / \overline{\mathbb{Q}}$. Then the set $X_{(n-\dim X)}$ is of bdd. ht. ...

nope.

<u>Problem</u>: Unlikely intersections!

<u>Def</u>: $Y \subset X$ is an unlikely intersection with a proper coset $gH \subset \mathbb{G}_m^n$ if

$\operatorname{codim} Y \not\leq \operatorname{codim} gH + \operatorname{codim} X.$

<u>Ex</u>: Two surfaces in $\mathbb{G}_m^4$ intersecting in a curve:

$$X : (s,t) \mapsto (s, 1-s, t, c-t).$$

$$H = (1,1) \times \mathbb{G}_m^2 \leq \mathbb{G}_m^4.$$

Cosets: $(a, 1-a) \times \mathbb{G}_m^2$

$$\cap\ X$$

$=$ a curve given by

$$t \mapsto (a, 1-a, t, c-t)!$$

<u>Def</u>: $X^{oa} = X - $ (union of all unlikely $\cap$'s).

Thm **(Habegger, '09):**

$X \subset \mathbb{G}_m^n$ irreducible / $\overline{\mathbb{Q}}$.

Then the Weil ht. of pts. in

$$X^{oa} \cap X_{(n-\dim X)}$$

is bdd.

Q: How likely are unlikely intersections?

Theorem 1.9 **(BMZ, '07):**

The maximal unlikely $\cap$'s of +ve dim. come from $\cap$'s w/ cosets $cH_1, \ldots, cH_n$.

$\Rightarrow X^{oa}$ is Zariski open in X.

Finally:

Zilber Conj:

$X \subset \mathbb{G}_m^n / \mathbb{C}$, then

$\exists\ \Omega_X = \{$ finite set of torsion cosets $\}$

s.t. "all unlikely $\cap$'s of X w/ torsion cosets come from Ω_X", i.e.:

- If K is a torsion coset s.t. $Y \subset X \cap K$ satisfies $\dim Y > \dim X + \dim K - n$,

then $Y \leq T$ for some $T \in \Omega_X$.

Ex. $Y = \{pt.\}$, then

Zilber: $X_{(n - \dim X - 1)}$ is contained in a finite union of proper alg. subgps.