ALGEBRA TERM TEST

Justify all your answers with careful proofs.

(1) Let p be a prime and G a p-group. Let X be a finite set with a G-action, and let

$$n = \#\{x \in X | g \cdot x = x \text{ for all } g \in G\}.$$

Show that $n \equiv \#X \mod p$.

(2) (a) What is the maximal order of an element in S_7? Write down an example of an element with maximal order.

(b) Consider the element $\sigma := (123)(456) \in S_6$. What is the size of its conjugacy class? What is the size of its centralizer, that is, $\{g \in G | g\sigma g^{-1} = \sigma\}$?

(3) Let $D_{2n} = \langle \sigma, \tau | \sigma^n, \tau^2, \tau\sigma\tau^{-1}\sigma \rangle$ be the dihedral group with $2n$ elements. Recall that the commutator subgroup $[D_{2n}, D_{2n}]$ is defined to be the (normal) subgroup generated by elements of the form $ghg^{-1}h^{-1}, g, h \in D_{2n}$. The *abelianization* of D_{2n} is $D_{2n}/[D_{2n}, D_{2n}]$. What is the order of the abelianization? What is its group structure? (Hint: the answer depends on the parity of n.)

(4) Show that there are no finite simple groups of order 30. Hint: How many subgroups/elements of order 3 are there? How many subgroups/elements of order 5?