ALGEBRA TERM TEST

Justify all your answers with careful proofs.

(1) Let p be a prime and G a p-group. Let X be a finite set with a G-action, and let

$$n = \#\{x \in X | g \cdot x = x \text{ for all } g \in G\}.$$

Show that $n \equiv \#X \mod p$.

Proof. Any non-trivial G-orbit $G \cdot x$ in X is isomorphic (by the orbit-stabilizer theorem) to $G/\mathrm{Stab}_G(x)$, which has order divisible by as G is a p-group. Writing X as the union of the fixed points with the non-trivial orbits, we see that

$$\#X = n + \sum_{\text{orbits } G \cdot x} \#G \cdot x,$$

and reducing this equality mod p gives the desired result. □

(2) (a) What is the maximal order of an element in S_7? Write down an example of an element with maximal order.

Proof. The maximal order is 12, realized by e.g. (1234)(567). It follows from the fact that disjoint cycles commute that the order of an element with cycle type $(n_1, \cdots, n_r)$ is the lcm of the n_i; now exhaustive search shows that 12 is the maximum lcm of a partition of 7. □

(b) Consider the element $\sigma := (123)(456) \in S_6$. What is the size of its conjugacy class? What is the size of its centralizer, that is, $\{g \in G | g\sigma g^{-1} = \sigma\}$?

Proof. To compute the size of the conjugacy class, note that we may put the six numbers in order in 6! ways. We get the same element of S_n if we cyclically permute the first 3 or last 3 numbers, or if we swap the first three with the last three. This gives that the conjugacy class has size $6!/(3 \cdot 3 \cdot 2) = 40$. The centralizer is exactly the stabilizer of the element under conjugation, so orbit-stabilizer gives that it has size $6!/40 = 18$. □

(3) Let $D_{2n} = \langle \sigma, \tau | \sigma^n, \tau^2, \tau\sigma\tau^{-1}\sigma \rangle$ be the dihedral group with $2n$ elements. Recall that the commutator subgroup $[D_{2n}, D_{2n}]$ is defined to be the (normal) subgroup generated by elements of the form $ghg^{-1}h^{-1}, g, h \in D_{2n}$. The *abelianization* of D_{2n} is $D_{2n}/[D_{2n}, D_{2n}]$.

What is the order of the abelianization? What is its group structure? (Hint: the answer depends on the parity of n.)

Proof. We first compute that $\tau\sigma\tau^{-1}\sigma^{-1} = \sigma^{-2}$ is in the commutator subgroup; hence the same is true for σ^2.

Case 1: n is odd. Then there exists m with $2m \equiv 1 \bmod n$, hence in this case $\sigma = (\sigma^2)^m$. Thus the commutator subgroup contains σ. But $D_{2n}/\langle\sigma\rangle = \langle\sigma,\tau|\sigma,\sigma^n,\tau^2,\tau\sigma\tau^{-1}\sigma\rangle = \langle\tau|\tau^2\rangle = \mathbb{Z}/2\mathbb{Z}$ is abelian, hence must in fact be the abelianization. It has order 2.

Case 2: $n = 2m$ is even. In this case we compute $D_{2n}/\langle\sigma^2\rangle = \langle\sigma,\tau|\sigma^2,\sigma^{2m},\tau^2,\tau\sigma\tau^{-1}\sigma\rangle = \langle\sigma,\tau|\sigma^2,\tau^2,\tau\sigma\tau^{-1}\sigma^{-1}\rangle = (\mathbb{Z}/2\mathbb{Z})^2$, which has order 4. (Here the isomorphism between the group with this presentation and $(\mathbb{Z}/2\mathbb{Z})^2$ sends σ to $(1,0)$ and τ to $(0,1)$.) □

(4) Show that there are no finite simple groups of order 30. Hint: How many subgroups/elements of order 3 are there? How many subgroups/elements of order 5?

Proof. 30 factors as $2\cdot 3\cdot 5$. Let us compute the numbers n_3, n_5 of 3-Sylow (resp. 5-Sylow) subgroups. We have $n_3 = 1 \bmod 3, n_3|10$, so $n_3 = 1$ or $n_3 = 10$. Similarly $n_5 = 1$ or $n_5 = 6$. If either n_3 or n_5 equals 1, we're done (as the relevant p-Sylow is normal), so assume for the sake of contradiction that $n_3 = 10, n_5 = 6$.

In this case there are 10 3-Sylows, each containing 2 elements of order 3, with pairwise trivial intersection (as they are simple). Hence there are $20 = 2\times 10$ elements of order 3. The same reasoning shows there are $24 = 4\cdot 6$ elements of order 5. But $20 + 24 > 30$, contradiction. □