ALGEBRA HW5

All rings are commutative in this problem set.

(1) (a) Let R be a ring and $I \subset R$ an ideal. Let M be an R-module. Construct an isomorphism $M \otimes_R (R/I) \xrightarrow{\sim} M/IM$, where $IM \subset M$ is the submodule generated by elements of the form $i \cdot m$, where $i \in I, m \in M$.
   (b) Deduce that for ideals $I, J \subset R$, there is a natural isomorphism
$$R/I \otimes_R R/J \xrightarrow{\sim} R/(I+J).$$
(2) (a) Let $A_1, \cdots, A_n$ be finite cyclic groups. Determine (with proof) the order of
$$A_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A_n.$$
   (b) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}$ as rings.

(3) Suppose R is a domain and M an R-module. Recall that
$$M_{\text{tor}} = \{x \in M \mid rx = 0 \text{ for some } r \in R - \{0\}\},$$
a submodule of M. We say that M is torsion if $M_{\text{tor}} = M$.
   (a) Let $\text{Ann}_R(M) := \{r \in R \mid rm = 0 \text{ for all } m \in M\}$, the annihilator of M. Show that $\text{Ann}_R(M)$ is an ideal of R.
   (b) Suppose that I, J are ideals of R such that $R/I \simeq R/J$ as R-modules. Show that $I = J$. (Hint: consider annihilators.)
   (c) Aside (for R any commutative ring): show that an R-module M is simple (i.e. it's nonzero and its only submodules are $\{0\}$ and M) if and only if M is isomorphic to R/I with I a maximal ideal of R.
   (d) If M is a finitely generated torsion R-module show that $\text{Ann}_R(M) \neq 0$.
   (e) Give an example of a domain R and a torsion R-module M such that $\text{Ann}_R(M) = 0$.

(4) Let R be an integral domain, M an R-module, and $K = \text{Frac}(R)$ the field of fractions of R. Show that M is torsion if and only of $M \otimes_R K = 0$.

(5) Let R be an integral domain of characteristic zero, i.e. such that no nonzero multiple of 1 is equal to zero. Let M be an $n \times n$ matrix over R. Show that M is nilpotent (i.e. $M^N = 0$ for some $N \gg 0$) if and only if $\text{Tr}(M^r) = 0$ for all $r > 0$.

(6) In this exercise we will prove the Cayley-Hamilton theorem over commutative rings R: a square matrix satisfies its own characteristic polynomial.

   (a) Show that the statement is true for matrices in Jordan normal form. Deduce that it is true for arbitrary matrices over $\mathbb{C}$.
   (b) Consider the ring $S = \mathbb{Z}[X_{ij}]_{i,j=1,\cdots n}$. Let M be the matrix $M = (X_{ij})_{i,j=1,\cdots n}$. Show that the Cayley-Hamilton theorem is true for this matrix. (Hint: Embed S in $\mathbb{C}$.)
   (c) Deduce that the Cayley-Hamilton theorem holds for arbitrary square matrices over arbitrary commutative rings R. (Hint: Given a matrix $M = (m_{ij})$, consider the map $\mathbb{Z}[X_{ij}]_{i,j=1,\cdots n} \to R$ sending X_{ij} to m_{ij}.)

(7) Prove that if k is a field, then $k[[x]]$ is a PID.