ALGEBRA HW4

(1) (a) Let k be a field. Show that $k[t]$ is a PID.
  (b) Suppose k is algebraically closed. Show that every maximal ideal of $k[t]$ is of the form $(t-a)$ for some $a \in k$.
  (c) Show that $\mathbb{C}[x,y]$ is not a PID.

(2) Let R be a PID. Show that every finitely generated R-module is isomorphic to
$$\operatorname{coker}(R^{\oplus a} \xrightarrow{\varphi} R^{\oplus b})$$
for some nonnnegative integers a, b and some *injective* map φ.

(3) Give an example of an integral domain R and a submodule of a free R-module which is not free.

(4) Let $H_{\mathbb{C}}$ be the ring
$$\mathbb{C}\langle i,j,k\rangle/(i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ki = j).$$
Write down a ring isomorphism between $H_{\mathbb{C}}$ and $\mathrm{Mat}_{2\times 2}(\mathbb{C})$.

(5) Let R be the ring given as the quotient of the (non-commutative) polynomial ring $\mathbb{C}\langle E,F,H\rangle$ by the 2-sided ideal
$$I = (EF - FE - H, HF - FH + 2F, HE - EH - 2E).$$
Let M be a left R-module which is finite-dimensional as a $\mathbb{C}$-vector space. Show that E, F act nilpotently on M, and that the eigenvalues of the H-action are integers. (Hint: think about the eigenvalues of H first.)

(6) Let R be a (commutative) Noetherian ring. Show that $R[t]$ is Noetherian. (Hint: Let $I \subset R[t]$ be an ideal and consider the ideal generated by the leading coefficients of the polynomials in I. Take polynomials whose leading coefficients generate this ideal; these will let you decrease the degree of polynomials in I of sufficiently large degree. Now throw in some more low-degree elements of I to take care of polynomials of lower degree.)

(7) Let R be a commutative ring. Show that the intersection of all prime ideals of R is the *nilradical*, namely
$$\mathcal{N} = \{x \mid \exists n \text{ s.t. } x^n = 0\}.$$
Hint: Show that a nilpotent element is contained in every prime ideal; this proves one inclusion. For the other, consider x which is not nilpotent and use Zorn's lemma to produce a maximal, hence prime, ideal not containing x.