ALGEBRA HW3

(1) Let $p \neq 2$ be a prime, and let $k > 0$ be a positive integer such that $2^k p > p!$. Show that no group of order $2^k p$ is simple.
(2) Show that the group of upper-triangular matrices in $GL_n(\mathbb{F}_p)$ is solvable.
(3) Let $G \subset GL_n(\mathbb{Z}/p\mathbb{Z}) = \mathrm{Aut}((\mathbb{Z}/p\mathbb{Z})^n)$ be a p-group. Show that there exists a non-zero element of $(\mathbb{Z}/p\mathbb{Z})^n$ fixed by G. (Hint: consider the number of fixed elements, modulo p.)
(4) Write down a careful, detailed proof of the Yoneda lemma.
(5) Let R be a ring.
  (a) Let $a, b \in R$, such that a, b commute. Let p be a prime number such that $p = 0$ in R. Show that $(a + b)^p = a^p + b^p$.
  (b) Let $x \in R$ be nilpotent, i.e. there exists some n such that $x^n = 0$. Show that $1 + x \in R^\times$, i.e. it has a multiplicative inverse.
  (c) Suppose that there exists a prime p such that $p = 0$ in R. Let $x \in R$ be nilpotent. Show that there exists some $m > 0$ with $(1 + x)^m = 1$.
(6) Consider the functor Rings $\to$ Grp sending R to $R^\times$, the group of elements of R with a multiplicative inverse. Show that this functor has a left adjoint.
(7) Show that the group $\langle a, b, c | a^2, b^3, c^4, abc \rangle$ is isomorphic to S_4 (by writing down an explicit isomorphism).