ALGEBRA HW2

(1) Let p, q be distinct primes. Show that any group G of order pq is a semidirect product of two cyclic groups.

(2) Let $SL_2(\mathbb{Z})$ be the group of 2×2 matrices with integer entries and determinant 1, under multiplication, i.e.

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Show that $SL_2(\mathbb{Z})$ is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(This is exercise 6.10 in Aluffi; you can check there if you need a hint.)

(3) Let $H \subset G$ be a subgroup of index 2 (i.e. $|G/H| = 2$). Show that H is normal. (Note that G need not be finite!) Give an example to show that subgroups of index 3 need not be normal.

(4) Construct a left adjoint to the inclusion $\mathrm{Ab} \hookrightarrow \mathrm{Grp}$.

(5) Find all normal subgroups of S_4 (with proof). Prove that S_4 is solvable.

(6) (a) Let G act on a set X. Suppose $x, y \in X$ are in the same G-orbit. Show that Stab_x is conjugate to Stab_y.
    (b) Let X, Y be transitive (left) G-sets. Choose $x \in X, y \in Y$. Show that X, Y are isomorphic as G-sets if and only if Stab_x is conjugate to Stab_y.

(7) Suppose $G = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_m \rangle$ with $m < n$. Show that G is infinite. (Hint: construct a nontrivial map $G \to \mathbb{Q}$. You may find (4) above helpful.)