ALGEBRA HW1

(1) Let G, H be groups. Recall that we may regard G, H as groupoids with one object, say $\mathcal{C}_G$ and $\mathcal{C}_H$. In this language, a group homomorphism $f : G \to H$ is the same as a functor $F : \mathcal{C}_G \to \mathcal{C}_H$.

Given two functors $A, B : \mathcal{C}_G \to \mathcal{C}_H$ (equivalently, group homomorphisms $a, b : G \to H$) let $\eta : A \to B$ be a natural transformation. Give a simple description of what this means in group-theoretic language—that is, how is a related to b?

(2) Look up adjoint functors on Wikipedia. Let $F : \mathrm{Grp} \to \mathrm{Sets}$ be the "forgetful" functor, sending a group $(G, \cdot, e)$ to its underlying set G. Construct (with proof) a left adjoint to F.

(3) Let G be a group. Show that the following are equivalent:
   (a) G is commutative.
   (b) The inversion map
$$\iota : G \to G$$
$$g \mapsto g^{-1}$$
   is a group homomorphism.

(4) Let p be a prime number.
   (a) Find the order of the group $GL_2(\mathbb{Z}/p\mathbb{Z})$.
   (b) Write down elements of $GL_2(\mathbb{Z}/p\mathbb{Z})$ of order p. How many such elements are there?
   (c) Write down an element of $GL_2(\mathbb{Z}/p\mathbb{Z})$ of order 2. Make sure your answer works if $p = 2$.
   (d) *Write down an element of order $p + 1$.

(5) Let p be a prime number. How many subgroups of S_p have order p? Show that they are all conjugate to each other.

(6) Let G be a group; we will study $\mathrm{Aut}(G)$.
   (a) Show that there is a natural group homomorphism
$$G \to \mathrm{Aut}(G)$$
$$g \mapsto (x \mapsto gxg^{-1}).$$
   The image of this map is, by definition, the group of *inner automorphisms* $\mathrm{Inn}(G)$.
   (b) Construct a natural isomorphism $G/Z(G) \xrightarrow{\sim} \mathrm{Inn}(G)$.
   (c) Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
   (d) The group $\mathrm{Out}(G)$ is defined to be $\mathrm{Aut}(G)/\mathrm{Inn}(G)$. Give an example, with proof, of a finite group such that $\mathrm{Out}(G)$ is non-trivial.

(7) Let $\sigma \in S_n$ have cycle type $(n_1, \cdots, n_r)$ (so that $\sum_i n_i = n$). What is the order of σ? What is the least n such that S_n contains an element of order 18? Give an example of such an element.