Étale cohomology

Last time: • started fppf descent.

Goal: Show qcoh sheaves/rep'ble functors give sheaves on $X_{ét}$, X_{fppf}

Thm (fppf descent for quasi-coherent sheaves)

$f : U \to X$ fppf cover

f^* induces an equivalence of categories

$$\mathrm{QCoh}(X) \xrightarrow{f^*} \{\text{descent data for quasi-coherent sheaves on } U/X\}.$$

Pf (1) Full faithfullness:

$$\begin{tikzcd} U \times_X U \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & \\ U \arrow[r, "f"] & X \end{tikzcd}$$

$\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{QCoh}(X)$

Want:

$$\begin{tikzcd} \mathrm{Hom}_X(\mathcal{F}_1, \mathcal{F}_2) \arrow[r, "f^*"] & \mathrm{Hom}_U(f^*\mathcal{F}_1, f^*\mathcal{F}_2) \arrow[r, "\pi_1^*"] \arrow[dr, "\pi_2^*"'] & \mathrm{Hom}_{U \times_X U}(\pi_1^* f^* \mathcal{F}_1, \pi_1^* f^* \mathcal{F}_2) \arrow[d, equal] \\ & & \mathrm{Hom}_{U \times_X U}(\pi_2^* f^* \mathcal{F}_1, \pi_2^* f^* \mathcal{F}_2) \end{tikzcd}$$

equalizer diagram

Lemma $R \to S$ faithfully flat ring map

$N \in R\text{-mod}$

$$\begin{array}{ccccc} & & n\otimes s \mapsto n\otimes 1\otimes s & & \\ N & \to & N\otimes_R S & \rightrightarrows & N\otimes_R S\otimes_R S \\ n\mapsto n\otimes 1 & & n\otimes s\mapsto n\otimes s\otimes 1 & & \end{array}$$ (proved last time).

is an equalizer diagram.

<u>Pf of full faithfullness</u> (1) Reduce to the case where $U \to X$ affine (exercise) - use that map is of finite presentation.

(2) $R \to S$ faithfully flat ($U = \operatorname{Spec} S$, $X = \operatorname{Spec} R$)
N, M - R-modules.

Want:

$$\operatorname{Hom}_R(M,N) \to \operatorname{Hom}_S(M\otimes S, N\otimes S) \rightrightarrows \operatorname{Hom}_{S\otimes S}(M\otimes S\otimes S, N\otimes S\otimes S)$$

to be equalizer diagram.

- Left exactness follows from injectivity of $N \to N\otimes S$
- exactness in the middle also comes from lemma.

<u>Cor</u> $\mathcal{F} \in QCoh(X)$
$\tilde{\mathcal{F}}^{ét} \in PreSh(X_{ét})$ $\quad \tilde{\mathcal{F}}^{ét}(U \xrightarrow{\pi} X) = \pi^*\mathcal{F}(U)$
is a sheaf on $X_{ét}$.

<u>Pf</u> Want: $U \to V$ étale cover,

$$\mathcal{F}(V) \to \mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_V U)$$

to be equalizer.

But this is the Hom equalizer diagram from before w/ $\mathcal{F}_1 = \mathcal{O}$, $\mathcal{F}_2 = \mathcal{F}$. $\square$

<u>Ex</u> $\mathcal{O}_X^{\acute{e}t} : (U \to X) \mapsto \Gamma(U, \mathcal{O}_U)$

Pf of fppf descent (part 2): essential surjectivity.

$U \xrightarrow{f} X$ fppf cover.

Given descent data $(\mathcal{F}, \varphi)$ on U/X.

Want: $\mathcal{G} \in QCoh(X)$ s.t.

$f^* \mathcal{G} \xrightarrow{\sim} \mathcal{F}$. (and descent data agrees).

Construction of $\mathcal{G}$:

(1) Reduce to the case of affine morphism (exercise)

$R \xrightarrow{f} S$, $M \in S\text{-mod}$

$\varphi : M \otimes_R S \to S \otimes_R M$ isom. of $S \otimes_R S$-module.

satisfying cocycle condition.

Set $K = eq\left(M \rightrightarrows S \otimes M\right)$, $m \mapsto 1 \otimes m$, $m \mapsto \varphi(m \otimes 1)$ (imagine $M = N \otimes S$, $N \in R\text{-mod}$)

<u>Claim</u> $K \otimes_R S \to M$ is an isom. (compatible w/ descent data)

$$R \to S \rightrightarrows S \otimes_R S$$

$$S \otimes_S M$$

$$K \to M \rightrightarrows M \otimes_R S$$

want to check that this map induces iso $K \otimes S \to M$.

(1) True if $R \to S$ has a section. (exercise)

$$\begin{tikzcd} U \arrow[d, shift right, "f"'] \\ X \arrow[u, shift right, "s"'] \end{tikzcd}$$

Given $\mathcal{F} \in QCoh(U)$ + descent data

Want: $\mathcal{G}$ s.t. $f^*\mathcal{G} \simeq \mathcal{F}$.

$\mathcal{G} \simeq s^*\mathcal{F}$.

(2) Have a map $K \to M$, want

$K \otimes S \to M$ is an iso.

After $- \otimes S$, $R \to S$ acquires a section.

$\Rightarrow K \otimes S \otimes S \to M \otimes S$ iso.

$\Rightarrow K \otimes S \to M$ iso. (b/c S is faithfully flat $/R$) $\blacksquare$

!!! ☺ !!!

$$X \xrightarrow{\text{Frob}} X^{(p)}_{/\mathbb{F}_p}$$ ← smooth variety

$$\mathrm{Vect}(X^{(p)}) \xrightarrow{\sim} \underset{\text{for v.b.'s}}{\text{Descent data}}(X/X^{(p)})$$

$$\downarrow$$

$\Big($v.b. $\mathcal{E}$ on X, w/ flat conn'n $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ s.t. ∇ has p-curvature zero$\Big)$

<u>Thm</u> $p : U \to X$ is a fppf cover.

Then $p^* : \mathrm{Sch}/X \longrightarrow$ Descent data for schemes on U/X

is fully faithful.

<u>Pf</u> (1) exercise – reduce to the case where <u>everything</u> is affine.

(enough to reduce to the case of AffSch/X)

(2)

Y, Z - X-schemes

$$\mathrm{Hom}_X(Y, Z) \to \mathrm{Hom}_U(p^*Y, p^*Z) \rightrightarrows \mathrm{Hom}_{U \times_X U}(\pi^*p^*Y, \pi^*p^*Z)$$

is an equalizer diagram.

WLOG: $Y = \underline{\mathrm{Spec}}_X \mathcal{O}_Y$, $Z = \underline{\mathrm{Spec}}_X \mathcal{O}_Z$

$\hookrightarrow$ qcoh sheaves of algebras.

$$\mathrm{Hom}_{\mathrm{qcohalg}(X)}(\mathcal{O}_Z, \mathcal{O}_Y) \to \mathrm{Hom}_U(p^*\mathcal{O}_Z, p^*\mathcal{O}_Y) \rightrightarrows \cdots$$

$\hookrightarrow$ want this to be equalizer diagram.

Follows from before. $\square$

__Cor__ If $Z \in \mathrm{Sch}/X$, then

$\mathrm{Hom}(-, Z)$ is a sheaf on X_{fppf}, $X_{\acute{E}t}$, $X_{\acute{e}t}$, ...

__Rem__ p^* is __not__ essentially surjective in general for schemes. (descent data for schemes relative to étale cover U/X = algebraic space)

It is ess. surjective for affine schemes

" for polarized schemes.

Ex $\mathbb{G}_m : U \mapsto \mathcal{O}_U(U)^\times$

$\mu_\ell : U \mapsto \{f \in \mathcal{O}_U(U) \mid f^\ell = 1\}$

$\underline{\mathbb{Z}/\ell} : U \longmapsto \mathrm{Hom}_{\mathrm{cont}}(U, \mathbb{Z}/\ell\mathbb{Z})$

$\mathrm{Hilb}^{p(t)}(\mathbb{P}^n)$

$\mathbb{P}^n$

are all sheaves.

Rem/ex Work out Galois descent from this PoV.

Cohomology

Goal: • The category of abelian sheaves on X_{et} is abelian.

• Enough injectives

Rem Both facts are true for the category of abelian sheaves on any site.

Crucial ingredient:

Thm τ is a site. The forgetful functor $Sh(\tau) \to Presh(\tau)$ has a left adjoint (called sheafification)

(We'll prove this for $\tau = X_{et}$).

Preliminaries

- Pushforward, pullback.

- Pushforwards

$f : \tau_1 \to \tau_2$ is a continuous morphism of sites.

(a functor $f^{-1} : \tau_2 \to \tau_1$ which preserves fiber products and sends covering families to covering families)

Defn Given $\mathcal{G} \in Sh(\tau_1)$

$f_* \mathcal{G} \in Sh(\tau_2)$ defined via

$$(f_* \mathcal{G})(U) := \mathcal{G}(f^{-1}(U))$$ ← usual formula for pushforwards.

Ex. $f_* \mathcal{G}$ is a sheaf.