Étale Cohomology – 9/17/2020

Last time:
- Čech-to-derived s.s.
- Mayer–Vietoris
- étale cohomology of qcoh sheaves
- étale coh. of $\mathbb{F}_p$ in char p.

Recall: $X/\mathbb{F}_p$ Artin-Schreier exact sequence of sheaves on $X_{ét}$.

$$0 \to \mathbb{F}_p \to \mathcal{O}_X^{ét} \xrightarrow{t \mapsto t^p - t} \mathcal{O}_X^{ét} \to 0$$

$$\cdots \to H^{i-1}(X, \mathcal{O}_X) \to H^i(X_{ét}, \mathbb{F}_p) \to H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_X) \to \cdots$$

X affine: $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$

$$0 \to \underset{\mathbb{F}_p^{\pi_0(X)}}{H^0(X, \mathbb{F}_p)} \to \mathcal{O}_X(X) \xrightarrow{t^p - t} \mathcal{O}_X(X) \to H^1(X, \mathbb{F}_p) \to 0$$

not f.g. in general

Rem $X/\mathbb{F}_p$ proper, $H^i(X_{ét}, \mathbb{F}_p)$ is f.d. by proper pushforward for coh. coh.

Ex E ell. curve$/k = \bar{k}$ of char p

$$H^1(E, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & E \text{ ordinary} \\ 0 & E \text{ s.s.} \end{cases}$$

Ex $H^i((\mathrm{Spec}\, k)_{et}, \mathcal{F})$

$Sh^{ab}(\mathrm{Spec}\, k_{et}) \overset{\iota}{\longleftrightarrow}$ discrete G-modules

(G = absolute Galois group of k)

Cor $H^i((\mathrm{Spec}\, k)_{et}, \mathcal{F}) \cong H^i(G, \iota\mathcal{F})$

Compare to Čech cohomology: $\check{C}(U/\mathrm{Spec}\, k, \mathcal{F})$

$U = \mathrm{Spec}\, L$: L/k is sep'ble field extn.

$$\check{C}(U/\mathrm{Spec}\, k, \mathcal{F}) = \mathcal{F}(U) \to \mathcal{F}(U \times U) \to \cdots$$

Assume L/k Galois w/ gp $G(L/k)$

$$\check{C}(U/\mathrm{Spec}\, k, \mathcal{F}) = \mathcal{F}(U) \to \mathcal{F}(G(L/k) \times U) \to \mathcal{F}(G(L/k)^2 \times U) \to \cdots$$

Claim Complex is the same as the standard complex computing $H^i(G(L/k), \mathcal{F}(U))$. (exercise)

Cor $\check{C}((\mathrm{Spec}\, k)_{et}, \mathcal{F})$ is q.i. to the usual complex computing Galois coh. $\square$

Q When can compute étale coh. as gp coh.?

A If your space is a $K(\pi,1)$ space.

Goal for next few classes: compute étale coh. of curves$/k=\bar{k}$

$$\boxed{H^i(C_{ét}, \mathbb{Z}/\ell^n\mathbb{Z})} \qquad \ell \neq \mathrm{char}\, k$$

Today: $i=0,1$. $\quad C$ conn'd

$$H^0(C_{ét}, \mathbb{Z}/\ell^n\mathbb{Z}) = \mathbb{Z}/\ell^n\mathbb{Z}$$

Interpretation of $H^1(X_{ét}, \mathcal{F})$ in terms of torsors.

Defn (G-torsor) $G \in Sh^{gp}(X_{ét})$

Idea: A sheaf $\mathcal{F} \in Sh^{sets}(X_{ét})$ w/ an action by G s.t. G acts simply transitively on every fiber.

Actual defn: A torsor is a sheaf $T \in Sh^{sets}(X)$ w/ an action $G \times T \xrightarrow{a} T$ s.t. $G \times T \xrightarrow{(a,\pi_2)} T \times T$ is an isomorphism.

<u>Rem</u> $T \times T \xrightarrow{\sim} G \times T$

(if you pull back to T, you a "trivial torsor")

<u>Ex</u> G is a G-torsor. (trivial torsor)

<u>Ex</u> G - finite gp, $\underline{G} \in Sh(X^{\acute{e}t})$ is the constant sheaf.

$\underline{G}$-torsor $\longleftrightarrow$ finite étale w/ Galois gp G

<u>Ex</u> $\mathbb{G}_m$ - $(U \longmapsto \mathcal{O}_U(U)^\times)$

$\mathbb{G}_m = \mathrm{Hom}(-, \operatorname{Spec} k[t, t^{-1}])$

<u>Ex</u> of $\mathbb{G}_m$-torsor : Line bundle minus 0-section

$$\mathcal{L} \rightsquigarrow \underbrace{\underline{\operatorname{Spec}}_X \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^{\otimes n}}$$

$\mathbb{G}_m$-torsor

<u>Ex</u> $G = \underline{GL_n}$.

<u>Claim</u> natural bij b/w GL_n-torsors and vector bundles of rk n.

vect. bdls $\qquad$ torsors

$\mathcal{E} \rightsquigarrow$ bundle of frames

$\mathrm{Isom}_{X_{\acute{e}t}}(\mathcal{O}^{\oplus n}, \mathcal{E})$

$$(T \times \mathcal{O}_X^{\oplus n})/\underline{GL_n} \rightsquigarrow T$$

diagonal action

$\underline{GL_n} = \underline{\mathrm{Aut}}(\mathcal{O}_X^{\oplus n})$

<u>Defn</u> A G-torsor T is <u>split</u> by a cover $U \to X$ if $T|_{U_{ét}}$ is isom. to $G|_{U_{ét}}$ (as a torsor). ("locally trivial")

<u>Rem</u> Suppose T is rep'ble, and $T \to X$ is a cover then T is split by T.

<u>Ex</u> G : finite étale gp scheme$/X$, T is a G-torsor split by some $U \to X$. Then

(1) T rep'ble

(2) T is split by T.

<u>Pf</u> (1)$\Rightarrow$(2): $\begin{matrix} T \times_X U \\ \downarrow \\ U \end{matrix}$ is a cover b/c finite étale + remark ✓.

(1): Observation: $T|_{U_{ét}}$ is rep'ble.

$\simeq G|_{U_{ét}}$.

(ii) effectivity of descent for affine schemes.

Prop $\{T \text{ G-torsor split by } U \to X\}/\sim$

$$\updownarrow \varphi$$

$\check{H}^1(U/X, G)$ ⟵ makes sense for $G \in Sh^{grps}(X_{\acute{e}t})$

Pf $T|_{U_{\acute{e}t}} \overset{\varphi}{\xrightarrow{\sim}} G|_{U_{\acute{e}t}}$ (as a torsor)

$$\begin{tikzcd} U \times_X U \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\ U \arrow[d] \\ X \end{tikzcd}
\qquad
\begin{tikzcd} \pi_1^* T \arrow[r, "\sim"] \arrow[d, "\pi_1^*\varphi"', "\wr"] & \pi_2^* T \arrow[d, "\pi_2^*\varphi", "\wr"'] \\ \pi_1^* G \arrow[r, "\sim", "\tau"'] & \pi_2^* G \end{tikzcd}$$

$\tau \in \Gamma(U \times_X U, G)$

$\check{C}^1(U/X, G)$

Claim The cocycle condition $\Rightarrow$ this elt is in $\ker d$.

(exercise)

Claim If $T_1 \simeq T_2$, cocycles differ by a cobdry.

Prop $\check{H}^1(\tau, \mathcal{F}) \xrightarrow{\sim} \{\text{locally trivial } \mathcal{F}\text{-torsors}\}$

τ any site $\quad \simeq H^1(\tau, \mathcal{F})$

(if $\mathcal{F}$ abelian)

Cor $\{\text{locally trivial } G\text{-torsors}\}/\simeq \;\simeq \check{H}^1(X_{\acute{e}t}, G)$.

Thm (Hilbert 90)

$$\check{H}^1(X_{Zar}, \underline{GL_n}) \simeq \check{H}^1(X_{\acute{e}t}, \underline{GL_n}) \to \check{H}^1(X_{fppf}, \underline{GL_n})$$

is a bijection.

Pf Need is that locally split torsors are the same:

Claim Locally split $\underline{GL_n}$-torsor is fppf descent for a v.b.

Done by fppf descent for v.b.s.