Étale Cohomology - 9/15/20

Čech cohomology:

$$\check{C}(U/X, \mathcal{F}) = \mathcal{F}(U) \to \mathcal{F}(U \times_X U) \to \mathcal{F}(U \times_X U \times_X U) \to \cdots$$

$$\check{C}(X_{ét}, \mathcal{F}) = \varinjlim_{U/X \leftarrow \text{cover of } X} \check{C}(U/X, \mathcal{F})$$

<u>Warning</u> $\check{H}^i(X_{ét}, \mathcal{F})$ is not in general isom. to derived functor coh.

<u>Thm</u> (Milne, EC, III) Čech cohomology is canonically isom. to derived functor coh. if X qc and satisfies:

any finite subset of X is contained in an affine (true if X is quasi-projective)

<u>Rem</u> Version of Čech coh. where covers are replaced by "hypercovers" does compute derived functor cohomology.

Čech-to-derived spectral sequence:

$$\underbrace{\tilde{\mathcal{F}} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots}_{\text{injective res'l'n of } \tilde{\mathcal{F}}.}$$

Given a cover $U \to X$, get

$$C^\bullet(U/X, \mathcal{I}^0) \to C^\bullet(U/X, \mathcal{I}^1) \to \cdots$$

Coh. in the horizontal direction, then vertical direction

E_2: $\check{H}^i(U, \mathcal{H}^j(\tilde{\mathcal{F}}))$ — presheaf: $V \mapsto H^j_{et}(V, \tilde{\mathcal{F}})$

$$\begin{array}{ccc} & & \vdots \\ \vdots & & q \\ j & & \\ C^1(U/X, \mathcal{I}^0) = \mathcal{I}^0(U \times_X U) & \to & \mathcal{I}^1(U \times_X U) \\ \uparrow & & \uparrow \\ C^0(U/X, \mathcal{I}^0) = \mathcal{I}^0(U) & \to & \mathcal{I}^1(U) \end{array}$$

Coh. in the vertical direction, then hor. direction:

E_2: $H^i(\Gamma(X, \mathcal{I})) = H^i(X, \tilde{\mathcal{F}})$

$= E_\infty$.

Get s.s.: $\check{H}^r(U, H^s(\tilde{\mathcal{F}})) \Rightarrow H^{r+s}(X_{et}, \tilde{\mathcal{F}})$

(Exercise: Last time, I claimed that if

$\check{C}^\bullet(X_{et}, -)$ is exact on $Sh^{ab}(X_{et})$

then $\check{H}^\bullet = H^\bullet$. Prove this using the

Čech to derived functor s.s.)

Mayer-Vietoris: $U = U_0 \cup U_1$

$U_i \subseteq U$ is a Zariski-open subset.

<u>Prop</u> $\exists$ functorial long exact sequence

$$\cdots \to H^s(U, \mathcal{F}) \xrightarrow{res} H^s(U_0, \mathcal{F}) \oplus H^s(U_1, \mathcal{F}) \xrightarrow{res} H^s(U_0 \cap U_1, \mathcal{F}) \xrightarrow{\delta} H^{s+1}(U, \mathcal{F}) \to \cdots$$

<u>Pf</u> Apply Čech-to-derived s.s. to the cover $U_0 \sqcup U_1 \to U$

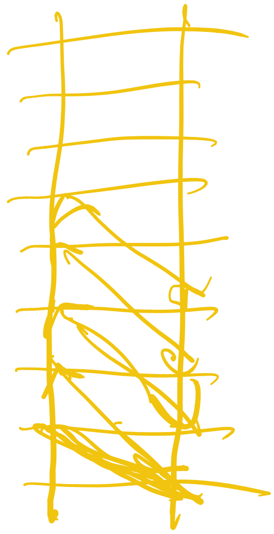
$$\mathcal{F}(U_0 \sqcup U_1) \to \mathcal{F}((U_0 \sqcup U_1)^{\times_U 2}) \to \mathcal{F}((U_0 \sqcup U_1)^{\times 3}) \to \cdots$$

<u>Claim</u>: q.i. to
(exercise).

$$\mathcal{F}(U_0 \sqcup U_1) \to \mathcal{F}(U_0 \cap U_1)$$
$$\mathcal{F}(U_0) \times \mathcal{F}(U_1)$$

(uses that this is a Zariski cover)

Given this: Čech to derived E_2 page vanishes except first 2 columns



this data is the same as a LES

Thm X scheme, $\mathcal{F} \in QCoh(X)$. Then

$$H^i(X, \mathcal{F}) = H^i(X_{ét}, \mathcal{F}^{ét}) = H^i(X_{fppf}, \mathcal{F}^{fppf})$$

usual (Zariski) coh. of $\mathcal{F}$ small site

Rem

$$\begin{array}{ccc} H^i(X_{Zar}, \mathcal{F}) & = & H^i(QCoh(X), \mathcal{F}) \\ \| & & \| \\ Ext^i_{Sh(X_{Zar})}(\mathbb{Z}, \mathcal{F}) & & Ext^i_{QCoh(X)}(\mathcal{O}_X, \mathcal{F}) \end{array}$$

(X qcqs)

<u>Pf</u> (if X is qc, seperated, Čech coh. computes derived functor cohomology).

(1) Every cover can be refined to a finite cover by affines.

(2) Suppose X affine, $U \to X$ is an fppf affine cover.

<u>Claim</u> $\check{C}(U/X, \mathcal{F})$ is exact if $\mathcal{F} = \widetilde{M}$ is quasi-coherent.

<u>Pf</u> $U = \operatorname{Spec} B$, $X = \operatorname{Spec} A$, $M \in A\text{-mod}$

$$M \to \underbrace{M \otimes B \to M \otimes B \otimes B \to M \otimes_A B^{\otimes 3} \to \cdots}_{\text{Amitsur complex.}}$$

This is exact by the same argument as in the proof of fppf descent

Hint: Base change to B, write down null-htpy using a section

<u>Claim</u> $\Rightarrow$ $\check{H}^i(U/X, \mathcal{F}) = \begin{cases} \mathcal{F}(X) & i = 0 \\ 0 & i > 0 \end{cases}$

for $\mathcal{F}$ qcoh, U, X affine.

(3) $\check{H}^i(X_{et}, \mathcal{F}^{et}) = \begin{cases} \mathcal{F}(X) & i=0 \\ 0 & i>0 \end{cases}$

if X affine, $\mathcal{F} \in QCoh(X)$.

Pf Affine covers are cofinal in the diagram of covers.

(4) X qc, sep'd, Čech coh. computes derived functor coh:

Take an affine cover $\mathcal{U} \to X$, use Čech-to-derived s.s. (exercise)

Ex $X = \mathbb{P}^n$, $\mathcal{F} = \mathcal{O}_X$

$$H^i(\mathbb{P}^n_{et}, \mathcal{O}_X^{et}) = \begin{cases} k & \text{if } i=0 \\ 0 & \text{if } i>0 \end{cases}$$

Ex $X/\mathbb{F}_p$ is a quasi-proj. variety

$$H^i(X_{et}, \underline{\mathbb{F}_p}) = ???$$

$\mathbb{G}_a = Hom(-, \mathbb{A}^1)$, $\mathbb{G}_a(U) = \mathcal{O}_U(U)$

Claim $0 \to \mathbb{F}_p \to \mathbb{G}_a \xrightarrow{x^p - x} \mathbb{G}_a$ is exact

Pf True at the level of representing objects (as $f^p - f = 0 \Rightarrow f$ constant)

Claim $0 \to \mathbb{F}_p \to \mathbb{G}_a \to \mathbb{G}_a \to 0$ is exact

Pf Need: $\mathbb{G}_a \xrightarrow{x^p - x} \mathbb{G}_a$ is an epimorphism

e.g. Given $f \in \mathcal{O}_U(U) = \mathbb{G}_a(U)$, need to solve $x^p - x = f$ étale-locally on U.

$$\begin{tikzcd} \mathbb{G}_a \times_{\mathbb{G}_a} U \arrow[r] \arrow[d] & \mathbb{G}_a \arrow[d, "x^p - x"] \\ U \arrow[r, "f"] & \mathbb{G}_a \end{tikzcd}$$

étale cover

Cor LES

$$0 \to H^0(X_{\text{ét}}, \mathbb{F}_p) \to H^0(X_{\text{ét}}, \mathbb{G}_a) \xrightarrow{x^p - x} H^0(X, \mathcal{O}_X)$$

$$\shortparallel$$

$$H^0(X, \mathcal{O}_X)$$

$$H^1(X_{et}, \mathbb{F}_p) \to H^1(X, \mathcal{O}_X) \xrightarrow{x^p - x} H^1(X, \mathcal{O}_X)$$

$$\to H^2(X, \mathbb{F}_p) \to \cdots$$

Ex $X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{F}_p[t]$

$$0 \to \underset{\mathbb{F}_p}{\underset{\|}{H^0(\mathbb{A}^1_{et}, \mathbb{F}_p)}} \to \mathbb{F}_p[t] \xrightarrow{t \mapsto t^p - t} \mathbb{F}_p[t] \to H^1(\mathbb{A}^1_{et}, \mathbb{F}_p) \to 0$$

huge.