Étale cohomology

Last time: - Stalks, sheafification, $Sh(X_{ét})$ is abelian

Niklas's Q: X_{fppf} $X_{ét}$

$$\begin{array}{ccc} X_{fppf} & & X_{ét} \\ & \searrow & \downarrow \\ & & X_{ét,\,fin.pres.} \end{array}$$

$\downarrow$ ← induces an equivalence on $Sh(-)$.

Thm $Sh(X_{ét})$ has enough injectives.

Pf. $\mathcal{F} \in Sh^{ab}(X_{ét})$, want injective sheaf $\mathcal{I}$ w/ $\mathcal{F} \hookrightarrow \mathcal{I}$.

For each $x \in X$, choose a geom. pt $\bar{x} \to x \to X$.

Let $I(\bar{x})$ be an injective ab. gp w/ a map $\mathcal{F}_{\bar{x}} \to I(\bar{x})$.

Claim $\prod_{\bar{x}} (i_{\bar{x}})_* I(\bar{x}) =: \mathcal{I}$

Pf (1) $\mathcal{F} \hookrightarrow \mathcal{I}$ (3) $\mathcal{I}$ is injective. (exercise)

(2) Monic: check on stalks.

Achieved: $Sh(X_{ét})$ Abelian, w/ enough injectives

Rem True for $Sh(\tau)$, τ any site, but substantially harder.

Inverse image

$f: X \to Y$

(i) (Presheaf)

$f^{-1}: \mathrm{Pre}(Y_{\acute{e}t}) \to \mathrm{PreSh}(X_{\acute{e}t})$

$$f^{-1}\mathcal{F}(V \xrightarrow[\acute{e}t]{} X) = \varinjlim \mathcal{F}(U \to X)$$

where limit is over

$$\begin{array}{ccc} V & \to & U \\ \downarrow \acute{e}t & & \downarrow \acute{e}t \\ Y & \to & X \end{array}$$

<u>Fact</u> f^{-1} is left adjoint to pushforward

$$\mathrm{PreSh}(Y_{\acute{e}t}) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathrm{PreSh}(X_{\acute{e}t})$$

(ii) (Sheaves) $f^*\mathcal{F} := (f^{-1}\mathcal{F})^a$

<u>Thm</u> f^* is left adjoint to f_*.

<u>Pf</u> Sheafification is a left adjoint.

<u>Ex</u>: $\bar{x} \overset{\iota}{\hookrightarrow} X$ (gen. pt.) $\quad \iota^*\mathcal{F} = \mathcal{F}_{\bar{x}}$.

- $Y \xrightarrow{f} X$, $f^* \underline{\mathbb{Z}/\ell\mathbb{Z}} = \underline{\mathbb{Z}/\ell\mathbb{Z}}$

- $Y \xrightarrow{f} X$, $\mathcal{F} = \underline{\mathrm{Hom}}_X(-, Z)$

  $f^* \mathcal{F} = \underline{\mathrm{Hom}}_Y(-, Y \times_X Z)$.

"Compute" étale cohomology.

<u>Defn</u> Given $\mathcal{F} \in Sh(X_{ét})$

$$H^i(X_{ét}, \mathcal{F}) := R^i\Gamma(X, \mathcal{F}).$$

To compute: Choose: $\mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ ← injective objects

$$H^i(X_{ét}, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet))$$

$$(R^i\pi_*)\mathcal{F} = H^i(\pi_*\mathcal{I}^\bullet).$$ ← sheaves on $Y_{ét}$.

$\pi : X_{ét} \to Y_{ét}$

$L^i\pi^*\mathcal{G} = 0$ if $i > 0$.

<u>Claim</u> (exercise) Pullback is exact.

Basic Properties:

(1) $H^0(X_{ét}, \mathcal{F}) = \mathcal{F}(X) = \Gamma(X, \mathcal{F})$

(2) $H^i(\mathcal{I}) = 0$ for $i > 0$, $\mathcal{I}$ injective

(3) $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ SES in $Sh(X_{ét})$

get $\to H^{i-1}(X_{ét}, \mathcal{F}_3) \to H^i(X_{ét}, \mathcal{F}_1) \to H^i(X_{ét}, \mathcal{F}_2) \to H^i(X_{ét}, \mathcal{F}_3) \to \cdots$

<u>Ex</u> k-field, $Sh((\operatorname{Spec} k)_{ét})$ (Choose a sep'bl closure k^s of k)

$G = \operatorname{Gal}(k^s/k)$

<u>Claim</u> $Sh((\operatorname{Spec} k)_{ét}) \xrightarrow{c}$ Discrete G-modules

$$\mathcal{F} \longmapsto \varinjlim_{k \subset L \subset k^s} \mathcal{F}(\operatorname{Spec} L)$$

is an equivalence of categories.

<u>Cor</u> $H^i((\operatorname{Spec} k)_{ét}, \mathcal{F}) = H^i(G, c\mathcal{F})$

<u>Pf of claim</u> Inverse functor:

Given $V \to \operatorname{Spec} k$ étale, $V = \bigsqcup_{k'/k \text{ fin sep'ble}} \operatorname{Spec} k'$

Given a discrete G-module M,
$M \mapsto (V \mapsto \Gamma(M^{Gal(k^s/k')}))$.

<u>Pf of Cor</u> $\Gamma(\operatorname{Spec} k, \mathcal{F}) = (\mathcal{L}\mathcal{F})^G$

$H^0 \overset{\sim}{\longleftrightarrow}$ invts

$\Rightarrow$ étale cohomology = group cohomology.

<u>Ex</u> Discrete G-modules $\leftrightarrows Sh((\operatorname{Spec} k)_{ét})$
$E(k^s) \rightleftarrows \operatorname{Hom}(-, E)$
$\uparrow$ ell. curve

Čech Cohomology

(1) Čech coh. does not always compute étale cohomology

(2) Čech coh. is not actually computable (b/c in general acyclic covers DNE).

$\mathcal{U} = \bigcup_i U_i \to X$ étale cover. (Vork w/ $X_{ét}$, fin. pres.)

$\mathcal{F} \in Sh(X_{ét})$

$$X \leftarrow U \leftleftarrows U \times_X U \leftleftarrows U \times_X U \times_X U \leftleftarrows \cdots$$

$$\downarrow \mathcal{F}(-)$$

$$\mathcal{F}(U) \underset{d^1}{\overset{d^0}{\rightrightarrows}} \mathcal{F}(U \times_X U) \rightrightarrows \mathcal{F}(U \times_X U \times_X U) \cdots$$

$$\downarrow \textstyle\sum (-1)^i d^i$$

$$\check{C}^\bullet(U/X, \mathcal{F}) : 0 \to \mathcal{F}(U) \to \mathcal{F}(U \times_X U) \to \cdots$$

$$\check{C}^\bullet(X_{ét}, \mathcal{F}) := \varinjlim_{\substack{\{U_i \to X\} \\ \text{covering family}}} \check{C}^\bullet(U/X, \mathcal{F})$$

Defn $\check{H}^i(U/X, \mathcal{F}) = H^i(\check{C}^\bullet(U/X, \mathcal{F}))$

$\check{H}^i(X_{ét}, \mathcal{F}) = H^i(\check{C}^\bullet(X_{ét}, \mathcal{F}))$

Prop $\check{H}^0(U/X, \mathcal{F}) = \check{H}^0(X_{ét}, \mathcal{F}) = H^0(X_{ét}, \mathcal{F})$

Pf $\mathcal{F}(X) \to \mathcal{F}(U) \to \mathcal{F}(U \times_X U)$ exact

(sheaf condition).

Prop $\check{H}^i(U/X, I) = \check{H}^i(X_{ét}, I) = 0$

$i > 0$, $\mathcal{I}$ injective.

Pf Enough to show $C^\bullet(U/X, \mathcal{I})$ is exact away from 0.

(1) Let $\mathbb{Z}_U = \mathbb{Z}[\mathrm{Hom}_X(-, U)]$

Then (claim)

$$C^\bullet(U/X, \mathcal{I}) = \mathrm{Hom}(\mathbb{Z}_U, \mathcal{I})$$

$$\begin{array}{ccc}
\mathbb{Z}_U & & \mathrm{Hom}(\mathbb{Z}_U, \mathcal{I}) \\
\downarrow & & \downarrow \\
\mathbb{Z}_{U \times_X U} & \xrightarrow{\mathrm{Hom}(-, \mathcal{I})} & \mathrm{Hom}(\mathbb{Z}_{U \times_X U}, \mathcal{I}) \\
\downarrow & & \downarrow \\
\mathbb{Z}_{U \times_X U \times_X U} & & \mathrm{Hom}(\mathbb{Z}_{U \times_X U \times_X U}, \mathcal{I}) \\
\downarrow & & \downarrow \\
 & & \vdots
\end{array}$$

(2) ETS: $\mathbb{Z} \to \mathbb{Z}_U \to \mathbb{Z}_{U \times_X U} \to \cdots$ is exact

(3) Special case of: given a set S,

$$\mathbb{Z} \to \mathbb{Z}^S \to \mathbb{Z}^{S \times S} \to \mathbb{Z}^{S \times S \times S} \to \cdots$$

exact for any S.

Pf Base change to $\mathbb{Z}^S$ — null-htpy.

(exercise)

Thm If for all

SES: $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ in $Sh(X_{et})$

$0 \to \check{C}(X_{et}, \mathcal{F}_1) \to \check{C}(X_{et}, \mathcal{F}_2) \to \check{C}(X_{et}, \mathcal{F}_3) \to 0$

is exact, $\check{H}^i(X_{et}, \mathcal{F}) \cong H^i(X_{et}, \mathcal{F})$ for all i, $\mathcal{F}$.

Pf Universal δ-functors. (we'll see this "Čech-to-derived s.s.")

Thm (Milne, III) True if X qc and if any finite subset of X is contained in an affine. (e.g. quasi-projective)