

Étale cohomology - Descent

Last time: sites, sheaves, morphisms of sites

Today: Descent!!!

Reminder

Defn (Continuous map of sites) If $(C, \mathcal{T}_1), (D, \mathcal{T}_2)$.

A morphism $(C, \mathcal{T}_1) \rightarrow (D, \mathcal{T}_2)$ is

a functor $f^{-1}: D \rightarrow C$ s.t.

(i) f^{-1} preserves fiber products

(ii) f^{-1} sends covering families to covering families.

Ex X scheme

$$X_{\text{Fppf}} \rightarrow X_{\text{Ét}} \rightarrow X_{\text{ét}} \rightarrow X_{\text{zar}}$$

Rem on terminology - Grothendieck pre-topology vs. Grothendieck topology
- Topos-category of sheaves on a site.

Q(1) How to check if some thing is a sheaf on $X_{\text{ét}} / X_{\text{Fppf}}$.
(2) How to construct sheaves?

Thm (1) If Y is an X -scheme, the functor

$$Z \mapsto \text{Hom}_X(Z, Y)$$

is a sheaf on X_{fppt} (hence on $X_{\text{ét}}, X_{\text{ét}}, \dots$)

(2) $\bar{\mathcal{F}} \in \text{QCoh}(X)$.

$$\sum_{\substack{Z \\ \downarrow \\ X}} \mapsto \Gamma(Z, p^* \bar{\mathcal{F}})$$

is a sheaf on X_{fppt} (hence on $X_{\text{ét}}, X_{\text{ét}}$).

Defn Call the associated sheaf on $X_{\text{ét}}, \bar{\mathcal{F}}^{\text{ét}}$.

Start w/ (2).

$U = \sqcup U_i$ is an fppt cover of X .

Q • Suppose $\bar{\mathcal{F}} \in \text{QCoh}(U)$. When does it come from a quasi-coherent sheaf on X ?

More precise: What extra structure do you need to "descend" to $\text{QCoh}(X)$?

• Given $\bar{\mathcal{F}}_1, \bar{\mathcal{F}}_2 \in \text{QCoh}(X)$, $f: \bar{\mathcal{F}}_1|_U \rightarrow \bar{\mathcal{F}}_2|_U$ when does f come from X ?

Ex $U = \sqcup U_i \rightarrow X$ is a Zariski cover.

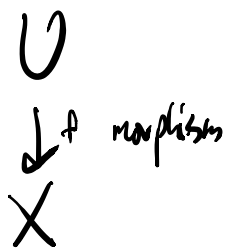
Data we need to get a sheaf on X is

$\bar{\sigma}_i \circ \nu_i \circ \nu_j \circ \bar{\sigma}_j \circ \bar{\sigma}_i \circ \nu_i \circ \nu_j$ "gluing data"

$\varphi_{j \circ \nu_i} \circ \varphi_{i \circ \nu_j} = \varphi_{i \circ \nu_i}$
 cocycle condition.

A morphism $\bar{\sigma} \rightarrow \mathcal{B}$ is the same as
 morphisms $\bar{\sigma}|_{U_i} \rightarrow \mathcal{B}|_{U_i}$ commuting w/ gluing
 data.

Defn (Descent data for qcsh sheaves)

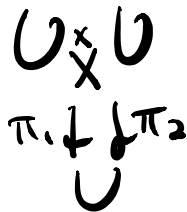


Descent data for a quasi-coherent
 sheaf on U/X is

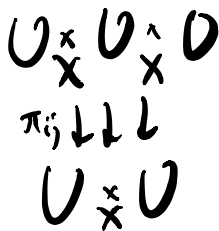
(1) $\bar{\sigma} \in \text{QCoh}(U)$.

(2) $\varphi: \pi_1^* \bar{\sigma} \xrightarrow{\sim} \pi_2^* \bar{\sigma}$

(3) $\pi_{23}^* \varphi \circ \pi_{12}^* \varphi = \pi_{13}^* \varphi$



Exercise: Check this agrees w/
 what we had before if $U \rightarrow X$
 is a Zariski cover.



Given descent data $(\bar{\sigma}, \varphi), (\mathcal{B}, \psi)$

a morphism $(\bar{\sigma}, \varphi) \rightarrow (\mathcal{B}, \psi)$ is
 a map $h: \bar{\sigma} \rightarrow \mathcal{B}$ s.t.



$$\pi_2^* \mathcal{F} \xrightarrow{\pi_1^* h} \pi_2^* \mathcal{G}$$

Thm (Descent for quasi-coherent sheaves)

Suppose $U \xrightarrow{f} X$ is fppt. Then f^* induces an equivalence of categories $\mathcal{L}_U \text{QCoh}(X)$ and descent data on U/X .

Explicit: Given $\mathcal{F} \in \text{QCoh}(X)$

$$\begin{array}{c} U \times U \\ \downarrow \pi_1 \quad \downarrow \pi_2 \\ U \\ \downarrow f \\ X \end{array}$$

$$f^* \mathcal{F} \in \text{QCoh}(U)$$

$$(f \circ \pi_1)^* \mathcal{F} \xrightarrow{\sim} (f \circ \pi_2)^* \mathcal{F}$$

is on $U \times U$.

$f \circ \pi_1 = f \circ \pi_2$ so pulling back it gives an iso

Ex (1) $U = \sqcup U_i$ - Zariski cover of X

$$\mathcal{O}_{U_i}^{\oplus n} \in \text{QCoh}(U_i)$$

To glue to a v.b. on X , need $\rho_{ij}: \mathcal{O}_{U_i \cap U_j}^{\oplus n} \xrightarrow{\sim} \mathcal{O}_{U_i \cap U_j}^{\oplus n}$

$$\text{s.t. } \rho_{ik} \circ \rho_{ij} = \rho_{jk} \text{ on } U_i \cap U_j \cap U_k$$

(2) L/k is a Galois extn w/ Galois gp G .

$\text{Spec } L \xrightarrow{f} \text{Spec } k$ Descent data on $\text{Spec } L / \text{Spec } k$

Spec k is a \mathbb{Q} -Coh sheaf on Spec L
 (L -vector space V)

$$\checkmark \text{ isom: } \pi_1^* V \xrightarrow{\varphi} \pi_2^* V$$

satisfying cocycle condition.

$$\text{Spec } L \times_{\text{Spec } k} \text{Spec } L = \text{Spec } L \otimes L = \bigsqcup_{L \rightarrow L} \text{Spec } L$$

$$= \bigsqcup_{\text{Gal}(L/k)} \text{Spec } L$$

(Exercise: convince yourself that descent data on this setting is the same as Galois descent data)

Pf of Thm

- (1) f^* is fully faithful
- (2) f^* is essentially surjective

(good reference: Ch. 6 of Néron models by BLR)

Pf of full faithfulness:

Given $\bar{\sigma}_1, \bar{\sigma}_2 \in \mathbb{Q}\text{-Coh}(X)$

$$\text{Hom}_X(\bar{\sigma}_1, \bar{\sigma}_2) \xrightarrow{f^*} \text{Hom}_U(f^* \bar{\sigma}_1, f^* \bar{\sigma}_2) \begin{array}{l} \xrightarrow{\pi_1^*} \text{Hom}_{U \times_U U}((f \circ \pi_1)^* \bar{\sigma}_1, (f \circ \pi_1)^* \bar{\sigma}_2) \\ \xrightarrow{\pi_2^*} \text{Hom}_{U \times_U U}((f \circ \pi_2)^* \bar{\sigma}_1, (f \circ \pi_2)^* \bar{\sigma}_2) \end{array}$$

Claim $g \in \text{Hom}_U(\mathcal{F}_1, \mathcal{F}_2)$

is a morphism of descent data if it maps to the same thing under π_1^*, π_2^* .
(exercise)

Full faithfulness is the same as the diagram above being an equalizer diagram.

Rem $\bar{\sigma}_1 = \mathcal{O}, \bar{\sigma}_2 = \bar{\sigma}$, this exactly shows $\bar{\sigma}^{\text{ét}}$ (resp $\bar{\sigma}^{\text{ppst}}$) is a sheaf.

Lemma $R \rightarrow S$ faithfully flat ring morphism.

$N \in R\text{-mod}$. Then

$$\begin{array}{ccc}
 N & \rightarrow & N \otimes_R S \xrightarrow{\text{id}_N \otimes \text{id}_S} N \otimes_R S \otimes_R S \\
 \downarrow n & \rightarrow & \downarrow n \otimes \text{id}_R \quad \downarrow \text{id}_N \otimes \text{id}_S
 \end{array}$$

is an equalizer diagram.

(Case Use $\bar{\sigma}_1 = \mathcal{O}, \bar{\sigma}_2 = \bar{\sigma}, U = \text{Spec } S, X = \text{Spec } R$)

Pf of lemma (1) WLOG $R \rightarrow S$ splits

$$\begin{array}{ccc}
 R & \rightarrow & S \\
 \downarrow & \nearrow m & \downarrow \text{id}_S \\
 S & \rightarrow & S \otimes_R S
 \end{array}$$

Claim we can replace R w/ S and S w/ $S \otimes_R S$.

Pf of claim We want to check

$$0 \rightarrow N \rightarrow N \otimes_R S \rightarrow N \otimes_R S \otimes_R S$$

↑ difference of 2 maps done

is exact. Suffices to do this after $-\otimes S$.

b/c a square of R -modules is exact iff its exact after $-\otimes S$.

(2) Suppose $R \xrightarrow{f} S$ splits. $\begin{matrix} \cup \\ \downarrow \\ \times \end{matrix}$

• $N \rightarrow N \otimes_R S$ injective b/c it has a splitting.
 \uparrow
 $\text{id}_N \otimes r$

• $\tilde{r}: S \otimes_R S \rightarrow S$
 $s_1 \otimes s_2 \mapsto s_1 \cdot f(r(s_2))$

$$\begin{aligned} \text{id}_N \otimes \tilde{r}(n \otimes s_1 - n \otimes 1 \otimes s) &= \\ n \otimes s - n \otimes f(r(s)) &= \\ n \otimes s - n \cdot r(s) \otimes 1 \end{aligned}$$

$$\Rightarrow n \otimes s \otimes 1 - n \otimes 1 \otimes s = 0 \quad (\text{kernel of differential})$$

$$n \otimes s - n \cdot r(s) \otimes 1 \in \text{image of differential}$$

(p.f. on free tensors \rightarrow do it in general). \square

Rem $R \rightarrow S$ faithfully flat

Amitsur complex

$$N \rightarrow N \circ S \rightarrow N \circ S \circ S \rightarrow \dots \rightarrow N \circ S \circ \dots \rightarrow \dots$$

Then This is exact.

Next time: Complete part of first descent.

$$\begin{array}{ccc} R & \rightarrow & S \\ \downarrow & & \downarrow \\ S & \rightarrow & S \circ S \end{array}$$