Étale cohomology - Descent

Last time: sites, sheaves, morphisms of sites

Today: Descent!!!

Reminder

Defn (Continuous map of sites) If $(C, \tau_1), (D, \tau_2)$.
A morphism $(C, T_1) \to (D, \tau_2)$ is
a functor $f^{-1}: D \to C$ s.t.
(i) f^{-1} preserves fiber products
(ii) f^{-1} sends covering families to covering families.

Ex X scheme

$$X_{Fppf} \to X_{Ét} \to X_{ét} \to X_{zar}$$

Rem on terminology - Grothendieck pre-topology vs. Grothendieck topology
- Topos - category of sheaves on a site.

Q (1) How to check if something is a sheaf on $X_{ét} / X_{fppf}$.
(2) How to construct sheaves?

Thm (1) If Y is an X-scheme, the functor
$$Z \longmapsto \mathrm{Hom}_X(Z, Y)$$
is a sheaf on X_{Fppf} (hence on $X_{\acute{E}t}$, $X_{\acute{e}t}$, ...)

(2) $\mathcal{F} \in QCoh(X)$.
$$\begin{array}{c} Z \\ \downarrow f \\ X \end{array} \longmapsto \Gamma(Z, f^*\mathcal{F})$$
is a sheaf on X_{Fppf} (hence on $X_{\acute{E}t}$, $X_{\acute{e}t}$).

Defn Call the associated sheaf on $X_{\acute{e}t}$, $\mathcal{F}^{\acute{e}t}$.

Start w/ (2).

$$\begin{array}{ll} U = \coprod U_i & \\ \downarrow \swarrow & \\ X & \end{array}$$ is an fppf cover of X.

Q: Suppose $\mathcal{F} \in QCoh(U)$. When does it come from a quasi-coherent sheaf on X?

More precise: What extra structure do you need to "descend it" to $QCoh(X)$.

- Given $\mathcal{F}_1, \mathcal{F}_2 \in QCoh(X)$, $f : \mathcal{F}_1|_U \to \mathcal{F}_2|_U$ when does f come from X?

Ex $U = \coprod U_i \to X$ is a Zariski cover.

Data we need to get a sheaf on X is

$\mathcal{F}_i|_{U_i \cap U_j} \xrightarrow[\varphi_{ij}]{\sim} \mathcal{F}_j|_{U_i \cap U_j}$ "gluing data"

$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$

- cocycle condition.

A morphism $\mathcal{F} \to \mathcal{G}$ is the same as morphisms $\mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ commuting w/ gluing data.

<u>Defn</u> (Descent data for qcoh sheaves)

$$\begin{array}{c} U \\ \downarrow f \text{ morphism} \\ X \end{array}$$

Descent data for a quasi-coherent sheaf on U/X is

(1) $\mathcal{F} \in QCoh(U)$.

(2) $\varphi : \pi_1^* \mathcal{F} \xrightarrow{\sim} \pi_2^* \mathcal{F}$

(3) $\pi_{23}^* \varphi \circ \pi_{12}^* \varphi = \pi_{13}^* \varphi$

$$\begin{array}{c} U \times_X U \\ \pi_1 \downarrow \ \downarrow \pi_2 \\ U \end{array}$$

$$\begin{array}{c} U \times_X U \times_X U \\ \pi_{ij} \downarrow \downarrow \downarrow \\ U \times_X U \end{array}$$

<u>Exercise</u>: Check this agrees w/ what we had before if $U \to X$ is a Zariski cover.

Given descent data $(\mathcal{F}, \varphi)$, $(\mathcal{G}, \psi)$ a morphism $(\mathcal{F}, \varphi) \to (\mathcal{G}, \psi)$ is a map $h : \mathcal{F} \to \mathcal{G}$ s.t.

$$\begin{array}{ccc} \pi_1^* \mathcal{F} & \xrightarrow{\pi_1^* h} & \pi_1^* \mathcal{G} \\ \downarrow \varphi & & \downarrow \psi \end{array}$$

commutes.

$$\pi_2^* \mathcal{F} \xrightarrow[\pi_2^* h]{} \pi_2^* \mathcal{G}$$

__Thm__ (Descent for quasi-coherent sheaves)

Suppose $U \xrightarrow{f} X$ is fppf. Then f^* induces an equivalence of categories b/w $QCoh(X)$ and descent data on U/X.

Explicit: Given $\mathcal{F} \in QCoh(X)$

$$\begin{tikzcd} U \times_X U \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\ U \arrow[d, "f"] \\ X \end{tikzcd}$$

$f^* \mathcal{F} \in QCoh(U)$

$(f \circ \pi_1)^* \mathcal{F} \xrightarrow{\sim} (f \circ \pi_2)^* \mathcal{F}$

iso on $U \times_X U$.

$f \circ \pi_1 = f \circ \pi_2$ so pulling back id gives an iso.

__Ex__ (1) $U = \bigsqcup U_i$ Zariski cover of X

$\mathcal{O}_{U_i}^{\oplus n} \in QCoh(U_i)$

To glue to a v.b. on X, need $\varphi_{ij} : \mathcal{O}_{U_i \cap U_j}^{\oplus n} \xrightarrow{\sim} \mathcal{O}_{U_i \cap U_j}^{\oplus n}$

s.t. $\varphi_{jk}|_{U_i \cap U_j \cap U_k} \circ \varphi_{ij}|_{U_i \cap U_j \cap U_k} = \varphi_{ik}|_{U_i \cap U_j \cap U_k}$

(2) L/k is a Galois extn w/ Galois gp G.

$\operatorname{Spec} L$ Descent data on $\operatorname{Spec} L / \operatorname{Spec} k$
$\downarrow$

Spec k is a QCoh sheaf on Spec L

(L-vector space V)

w/ isom: $\pi_1^* V \underset{\varphi}{\xrightarrow{\sim}} \pi_2^* V$

satisfying cocycle condition.

$$\operatorname{Spec} L \times_{\operatorname{Spec} k} \operatorname{Spec} L = \operatorname{Spec} L \otimes_k L = \coprod_{L \xrightarrow{\sim} L} \operatorname{Spec} L$$

$$= \coprod_{\mathrm{Gal}(L/k)} \operatorname{Spec} L$$

(Exercise: convince yourself that descent data in this setting is the same as Galois descent data)

Pf of Thm

(1) f^* is fully faithful

(2) f^* is essentially surjective

(good reference: Ch. 6 of Néron models by BLR)

Pf of full faithfulness:

Given $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{QCoh}(X)$

$$\operatorname{Hom}_X(\mathcal{F}_1, \mathcal{F}_2) \xrightarrow{f^*} \operatorname{Hom}_U(f^*\mathcal{F}_1, f^*\mathcal{F}_2) \underset{\pi_2^*}{\overset{\pi_1^*}{\rightrightarrows}} \begin{array}{c} \operatorname{Hom}_{U \times_X U}((f\circ\pi_1)^*\mathcal{F}_1, (f\circ\pi_1)^*\mathcal{F}_2) \\ \| \\ \operatorname{Hom}_{U \times_X U}((f\circ\pi_2)^*\mathcal{F}_1, (f\circ\pi_2)^*\mathcal{F}_2) \end{array}$$

<u>Claim</u> $g \in \mathrm{Hom}_U(f^*\mathcal{F}_1, f^*\mathcal{F}_2)$
is a morphism of descent data if it maps to the same thing under π_1^*, π_2^*.
(exercise)

Full faithfulness is the same as the diagram above being an equalizer diagram.

<u>Rem</u> $\mathcal{F}_1 = \mathcal{O}$, $\mathcal{F}_2 = \mathcal{F}$, this exactly shows $\mathcal{F}^{\acute{e}t}$ (resp $\mathcal{F}^{fppf}$) is a sheaf.

<u>Lemma</u> $R \to S$ faithfully flat ring morphism.
$N \in R\text{-mod}$. Then

$$\begin{tikzcd} N \arrow[r] & N \otimes_R S \arrow[r, shift left, "id_N \otimes id_S \otimes 1"] \arrow[r, shift right, "id_N \otimes 1 \otimes id_S"'] & N \otimes_R S \otimes_R S \end{tikzcd}$$
$n \longmapsto n \otimes 1$

is an equalizer diagram.

(Case where $\mathcal{F}_1 = \mathcal{O}$, $\mathcal{F}_2 = \tilde{N}$, $\begin{matrix} U = \mathrm{Spec}\, S \\ \downarrow \\ X = \mathrm{Spec}\, R \end{matrix}$)

<u>Pf of lemma</u> (1) WLOG $R \to S$ splits

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d, shift right, "1 \otimes id_S"'] \\ S \arrow[r] & S \otimes_R S \arrow[u, shift right, "m"'] \end{tikzcd}$$

<u>Claim</u> we can replace R w/ S and S w/ $S \otimes_R S$.

<u>Pf of claim</u> We want to check

$$0 \to N \to N \otimes_R S \xrightarrow[\substack{\text{difference} \\ \text{of 2 maps above}}]{} N \otimes_R S \otimes_R S$$

is exact. Suffices to do this after $-\otimes S$.
b/c a sequence of R-modules is exact iff its exact after $-\otimes S$.

(2) Suppose $R \underset{r}{\overset{f}{\rightleftarrows}} S$ splits.

$$\begin{matrix} U \\ \downarrow\uparrow \\ X \end{matrix}$$

- $N \xrightarrow{} N \otimes_R S$ injective b/c it has a splitting. $\quad \mathrm{id}_N \otimes r$

- $\tilde{r} : S \otimes_R S \to S$

$$s_1 \otimes s_2 \mapsto s_1 \cdot f(r(s_2))$$

$$\mathrm{id}_N \otimes \tilde{r}(n \otimes s \otimes 1 - n \otimes 1 \otimes s) =$$

$$n \otimes s - n \otimes f(r(s)) =$$

$$n \otimes s - n \cdot r(s) \otimes 1$$

$$\Rightarrow n \otimes s \otimes 1 - n \otimes 1 \otimes s = 0 \quad (\text{kernel of differential})$$

$$n \otimes s = n \cdot r(s) \otimes 1 \leftarrow \text{image of differential}$$

(pf on pure tensors — do it in general). $\square$

<u>Rem</u> $R \to S$ faithfully flat
Amitsur complex

$$N \to N \otimes S \to N \otimes S \otimes S \to \cdots \to N \otimes S^{\otimes r} \to \cdots$$

<u>Thm</u> This is exact.

Next time: Complete pf of fppf descent.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ S \arrow[r] & S \otimes S \end{tikzcd}$$