Étale Cohomology – 8/25/2020

Last time:

<u>Thm</u> (Serre) X sm. projective$/\mathbb{C}$

$[H] \in H^2(X, \mathbb{Z})$ hyperplane class

$F: X \to X$ s.t. $F^*[H] = q[H]$, $q \in \mathbb{Z}_{\geq 1}$

Then eigenvalues of F^* on $H^i(X, \mathbb{C})$ all have abs. value $= q^{i/2}$.

Office Hours: Mondays, 10am, Zoom link on course website.

HW: Join discord, post a fun fact about yourself

<u>Facts</u>: $L: H^i(X, \mathbb{C}) \to H^{i+2}(X, \mathbb{C})$

$$\alpha \longmapsto \alpha \cup [H]$$

Hard Lefschetz Thm: $H^j(X, \mathbb{C}) \cong \underline{\operatorname{im} L} \oplus H^j_{prim}$

$$H^j_{prim} = \bigoplus_{p+q=j} H^{p,q}_{prim}$$

Hodge index thm: $\alpha, \beta \in H^k(X)_{prim}$

$$\langle \alpha, \beta \rangle = i^k \int_X \alpha \wedge \overline{\beta} \wedge [H]^{n-k}$$

is definite on $H^{p,q}_{prim}$.

Pf of Serre's analogue of RH:

Want: eigenvalues of $F^* \curvearrowright H^k(X, \mathbb{C})$ have abs. $q^{k/2}$.

Suffices to do this for H^k_{prim}:

eigenvector $\alpha \in H^{k-2}(X, \mathbb{C})$, by induction

can assume eigenvalue has abs. value $q^{(k-2)/2}$.

$$F^*(\alpha \cup [H]) = F^*\alpha \cup F^*[H]$$
$$= \lambda\alpha \cup q[H]$$
$$= \underbrace{q\lambda}_{q^{k/2}}(\alpha \cup [H])$$

Let's do it for H^k_{prim}:

$\alpha \in H^k_{prim}$ be F^*-eigenvector w/ eigenvalue λ

$\alpha \in H^{p,q}_{prim}$ for some p, q s.t. $p+q=k$.

$$|\lambda|^2\langle\alpha,\alpha\rangle = \langle F^*\alpha, F^*\alpha\rangle$$
$$= i^k \int F^*\alpha \wedge F^*\bar{\alpha} \wedge [H]^{n-k}$$
$$= \frac{i^k}{q^{n-k}} \int F^*(\underbrace{\alpha \wedge \bar{\alpha} \wedge [H]^{n-k}}_{\in H^{2n}(X, \mathbb{C}),\ F^* \curvearrowright \text{ via } q^n})$$
$$= \frac{q^n i^k}{q^{n-k}} \int \alpha \wedge \bar{\alpha} \wedge [H]^{n-k}$$
$$= q^k \langle\alpha,\alpha\rangle$$

If $\langle \alpha, \alpha \rangle \neq 0$, get $|\lambda|^2 = q^k$
$\subset$ Hodge index thm. $\qquad |\lambda| = q^{k/2}$

Slogan: Structures on coh. $\Rightarrow$ RH.

Étale morphisms

$f : X \to Y$ morphism of schemes

__Defn__ f is étale if it is locally of finite presentation, flat, unramified.

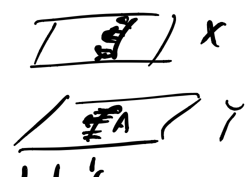
__Defn__ f is unramified if: $\Omega^1_{X/Y} = 0$
(equivalent: all residue field extns are sepble)

Equiv: • smooth of rel. dim'n 0
• lfp + formally étale

Defn (formally étale)

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

$I^n = 0$ for some n



Equiv: Locally "standard étale"

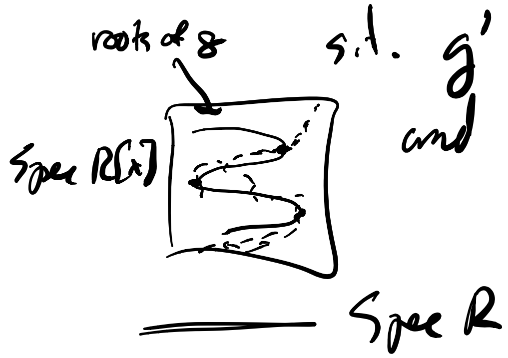
For each $x \in X$, $y = f(x)$, $\exists\, U \ni x$, $V \ni y$ s.t.

$$f(U) \subseteq V$$

$$V = \operatorname{Spec} R$$

$$U = \operatorname{Spec}\left(R[x]_h / g\right)$$

s.t. g' is a unit in $R[x]_h$ and g is monic.



slogan: g' a unit $\iff$ g has no double roots in fibers.

Exercise: Check that standard étale morphisms are étale.

<u>Examples</u>: <u>Ex</u> mult. by $[n]$ on ell. curve if n is invertible in the base.

<u>Ex</u> $\mathbb{G}_m = \operatorname{Spec} k[t,t^{-1}]$

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ t^n & \longleftarrow & t \end{array}$$ is étale if n prime to char k.

<u>Exercise</u> this is étale (Hint: $\frac{\partial}{\partial t}(t^n) = n\underbrace{t^{n-1}}_{\text{unit}}$)

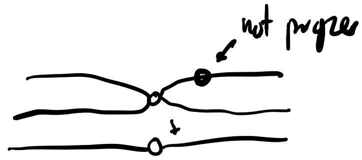
<u>Ex</u> $$\begin{array}{ccc} \mathbb{G}_m & \hookrightarrow & \mathbb{A}^1 \\ k[t,t^{-1}] & \longleftarrow & k[t] \end{array}$$

lfp ✓
flat ✓
$\Omega^1_{\mathbb{G}_m/\mathbb{A}^1} = 0$ ✓

<u>Prop</u> Any open immersion is étale.

<u>Ex</u> (An étale morphism which is not finite onto its image)

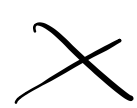
$$\begin{array}{ccc} \mathbb{G}_m \setminus \{1\} & \longrightarrow & \mathbb{G}_m \\ t^2 & \longleftarrow & t \end{array}$$ char $k \neq 2$



étale surjection but not finite étale.

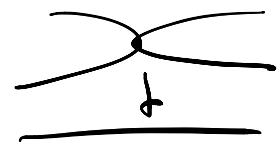
<u>Ex</u> finite sep'ble field extn.

<u>Non-ex</u>: • $X = \operatorname{Spec} k[x,y]/xy$



$\tilde{X} \to X$ not étale (not flat)

- $\begin{array}{ccc} \mathbb{A}^1 & \xrightarrow{f} & \mathbb{A}^1 \\ t^2 & \longleftarrow & t \end{array}$



not étale: $\Omega^1_f = k[t]\,dt \big/ d(t^2) = k[t]\,dt \big/ 2t\,dt$

$=$ supported at $t=0$
(if char $\neq 2$).

- finite flat, s.t. $\Omega^1_{X/Y}$ is not torsion?

$\begin{array}{ccc} \mathbb{A}^1 & \xrightarrow{F} & \mathbb{A}^1 \\ t^p & \longleftarrow & t \end{array}$ char p

$\Omega^1_F = k[t]\,dt \big/ d(t^p)$

$= k[t]\,dt.$

Ex $f : \mathbb{A}^m \xrightarrow{f_1,\dots,f_m} \mathbb{A}^m$

f is étale in a nbd of $(a_1,\dots,a_m)$ if

$\det\left(\frac{\partial f_i}{\partial x_j}\Big|_{(a_1,\dots,a_m)}\right)$ is a unit.

Prop. (1) open immersions are étale

(2) compositions of étale morphisms are étale
(hint: use cotangent exact sequence for $\Omega^1_{X/Y}$)

(3) base change of étale is étale

$$\begin{tikzcd} X \times_Y T \arrow[r] \arrow[d, "\text{étale}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\text{étale}"] \\ T \arrow[r] & Y \end{tikzcd}$$

(cf. 1(2 out of 3)) $\varphi \circ \psi$, φ étale
$\Rightarrow \psi$ étale (exercise)

Prop Étale morphisms on varieties over $k = \bar{k}$ induce isomorphisms on complete local rings at closed pts.

Pf (exercise) Hint: criterion for formal étaleness.

Cor (informal) any property that can be checked at level of complete local rings is true for source of étale morphism if it's true for target.

Sites - generalization of topological spaces/sheaf

Q What parts of the def'n of topological space do you need to define a sheaf?

(i) Open sets, inclusions
"category of open sets"

(presheaf on X: contravariant functor out of the category of open sets on X)

sheaf condition: · section to a sheaf is determined by its values on a cover
· glue sections which agree on intersections

(ii) collections of morphisms which are "covers"
(iii) existence of certain fiber products
(intersections)

Motivation: $U, V \subseteq X$

$U \times_X V = U \cap V.$

<u>Pre-Defn</u> (Grothendieck topology/site)
A category $\mathcal{C}$ w/ a collection of "covering families"
$\{X_\alpha \xrightarrow{f_\alpha} X\}_{\alpha \in A}$ s.t. ...
(I owe you the axioms)

<u>Ex</u> X top. space, $\mathcal{C}$ = category of open sets
$\{U_\alpha \to U\}$ is a covering family if U_α cover U.

<u>Ex</u> M-manifold. $\mathcal{C}$ = category of all $M' \xrightarrow{f} M$ s.t. f is locally on M' an isomorphism.

$\{M_\alpha \xrightarrow{f_\alpha} M\}$ is a covering family if $\bigcup \mathrm{im}(f_\alpha) = M$.

<u>Ex</u> X-scheme, $X_{ét}$ - category of all étale Y/X

$\{X_\alpha \xrightarrow{f_\alpha} X\}_\alpha$ a covering family if $\bigcup \mathrm{im}(f_\alpha) = X$.