Étale cohomology - 9/8/2020

Last time: fppf descent

Today: Cohomology

Remarks: • X -scheme

$$\mathrm{QCoh}(X_{zar}) \xrightarrow{\sim} \mathrm{QCoh}(X_{ét}) \xrightarrow{\sim} \mathrm{QCoh}(X_{fppf})$$

$$\mathrm{QCoh}(X_{Zar}) \simeq \mathrm{QCoh}(X_{\acute{E}t}) \simeq \mathrm{QCoh}(X_{fppf})$$

• Small correction: étale descent data for schemes $\rightsquigarrow$ alg. space

Goal: The category of abelian sheaves on $X_{ét}$ is abelian, w/ enough injectives.

Then $H^i(X_{ét}, \mathbb{Z}/\ell\mathbb{Z}) = R^i\Gamma(X_{ét}, \underline{\mathbb{Z}/\ell\mathbb{Z}})$

Crucial ingredient:

Thm τ a site. Then the forgetful

functor $Sh(\tau) \to PreSh(\tau)$ has a left adjoint. (Will prove for $X_{ét}$).
$\uparrow$ sheafification

(1) (Pushforward) $f: \tau_1 \to \tau_2$ is a cts map of sites.

For $\mathcal{G} \in Sh(\tau_1)$

<u>define</u> $f_* \mathcal{G} := \underset{\in \tau_2}{U} \mapsto \mathcal{G}(f^{-1}(U))$

(exercise: this is a sheaf).

<u>Ex.</u> $f: X \to Y$ map of schemes

$\rightsquigarrow$

$f: X_{ét} \to Y_{ét}$ $\qquad$ Get: $f_*: Sh(X_{ét}) \to Sh(Y_{ét})$.

$U \times_Y X / X \leftarrow U/Y$

<u>Ex</u> k alg. closed field

$\iota_{\bar{x}}: \operatorname{Spec} k \to X$.

$Sh((\operatorname{Spec} k)_{ét}) = \text{Sets}$ (exercise)

Given $\mathcal{F} \in Sh((\operatorname{Spec} k_{ét})) = \text{Sets}$

$$(\iota_{\bar{x}})_* \bar{\mathcal{F}}(U \to X) = \bar{\mathcal{F}}(U \times_X \bar{x})$$

$$= \bar{\mathcal{F}}\Big(\bigsqcup \operatorname{Spec} k\Big)$$

$\uparrow$ # of preimages of $\bar{x}$ in U

$$= \prod_{\#\text{preimages}} \bar{\mathcal{F}}(\operatorname{Spec} k).$$

<u>Defn</u> $(\iota_{\bar{x}})_* \bar{\mathcal{F}}$ is called a skyscraper sheaf.

(2) Pullbacks to a geometric pt

$$\iota_{\bar{x}} : \operatorname{Spec} k \to X \qquad k = \bar{k}.$$

$$\bar{\mathcal{F}}_{\bar{x}} = \iota_{\bar{x}}^* \bar{\mathcal{F}} \quad \text{for } \bar{\mathcal{F}} \in Sh(X_{ét}).$$

set

$\uparrow$ sheaf on $(\operatorname{Spec} k)_{ét}$

stalk of $\bar{\mathcal{F}}$ at $\bar{x}$

$$\bar{\mathcal{F}}_{\bar{x}} := \varinjlim_{(U, \bar{u})} \bar{\mathcal{F}}(U)$$

where direct limit is taken over diagrams

$$\begin{tikzcd} \bar{u} \arrow[r] \arrow[d] & U \arrow[d, "ét"] \\ \bar{x} \arrow[r, hook, "\iota_{\bar{x}}"] & X. \end{tikzcd}$$

$\leftarrow$ geom. pt.

Ex $\mathcal{F} = \underline{\mathbb{Z}/\ell\mathbb{Z}}$, $\bar{x} \hookrightarrow X$

$\iota_{\bar{x}}^{*} \underline{\mathbb{Z}/\ell\mathbb{Z}} = \mathbb{Z}/\ell\mathbb{Z}$

$\mathcal{F} = \mathcal{O}_X^{ét}$

$\iota_{\bar{x}}^{*} \mathcal{O}_X^{ét} = \mathcal{O}_{X,\bar{x}}^{sh}$ $\boxtimes$

Lemma $\mathcal{F}, \mathcal{G}$ sheaves of ab. gps on $X_{ét}$.

TFAE:

(i) $\mathcal{F} \to \mathcal{G}$ epimorphism

(ii) $\mathcal{F} \to \mathcal{G}$ locally surjective:

given $s \in \mathcal{G}(U)$, $\exists$ cover $U' \to U$ s.t. $s|_{U'}$ is in the image of some $s' \in \mathcal{F}(U')$.

(iii) $\mathcal{F}_{\bar{x}} \to \mathcal{G}_{\bar{x}}$ is surjective for all geom. pts $\bar{x} \to X$.

Pf (ii) $\Rightarrow$ (i) $\mathcal{F} \to \mathcal{G} \overset{a}{\underset{b}{\rightrightarrows}} \mathcal{H}$ want s.t. the 2 compositions agree, then $a = b$.

(i) $\Rightarrow$ (iii) Assume $\mathcal{F}_{\bar{x}} \to \mathcal{G}_{\bar{x}}$ is not surjective for some $\bar{x}$, w/ cokernel Λ.

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[rr, bend right, "0"'] & \mathcal{G} \arrow[r, "\neq 0"] & (\iota_{\bar{x}})_* \Lambda \end{tikzcd}$$

"the natural map"

↯.

(iii) ⇒ (ii) Given $s \in \mathcal{G}(U)$

want $U' \to U$ s.t. $s|_{U'}$ comes from $\mathcal{F}$.

Choose $\bar{x} \in U$. Know $\mathcal{F}_{\bar{x}} \to \mathcal{G}_{\bar{x}}$ is surjective:

$\exists$ some étale nbd of $\bar{x}$, $(V, \bar{v})$

s.t. $s|_V$ is in the image of $\mathcal{F}$.

Now choose $\bar{x}'$ not in image of V

and keep going.

$X \xrightarrow{\pi} \operatorname{Spec} k$, k alg. closed $\quad \pi_* = \Gamma$

$X \xrightarrow{\pi} \operatorname{Spec} k$: $\Gamma(\operatorname{Spec} k, \pi_* \mathcal{F}) = \Gamma(X, \mathcal{F})$

Lemma $0 \to \mathcal{F} \to \mathcal{G} \xrightarrow{f} \mathcal{K}$ is a sequence of abelian sheaves on $X_{ét}$. TFAE

(i) Sequence is exact : $\mathcal{F} = eq(\mathcal{G} \overset{s}{\underset{0}{\rightrightarrows}} \mathcal{H})$

(ii) $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is exact for all U

(iii) $0 \to \mathcal{F}_{\bar{x}} \to \mathcal{G}_{\bar{x}} \to \mathcal{H}_{\bar{x}}$ exact for all geom. pts $\bar{x}$.

<u>Pf</u> exercise. (<u>Claim</u>: Same pf as for top. spaces: (ii) ⇒ (iii) direct limits preserve exactness).

<u>Pf</u> (Sheafification exists for $PreSh(X_{\acute{e}t})$)

(i) Construct an "espace étale".

For each $x \in X$, choose a geom. pt $\bar{x}$ lying over x.

Given $\mathcal{F} \in PreSh(X_{\acute{e}t})$ define

$$Esp(\mathcal{F}) = \prod_{\bar{x}} (i_{\bar{x}})_* \mathcal{F}_{\bar{x}}.$$

sheaf.

Natural map of presheaves from $\mathcal{F} \to Esp(\mathcal{F})$

Defn $\mathcal{F}^a$ is the subsheaf of $Esp(\mathcal{F})$ "generated by $\mathcal{F}$".

$$\mathcal{F}^a(U) \subseteq Esp(\mathcal{F})(U)$$

$$"\{ s \in Esp(\mathcal{F})(U) \mid s \text{ is locally in the image of } \mathcal{F} \}"$$

Claim (1) $\mathcal{F}^a$ is a sheaf.

(2) Check that $\mathcal{F}^a$ is left adjoint to forgetful functor. (exercise)

Cor Colimits exist in $Sh(X_{\acute{e}t})$.

Pf (1) Colimits exist for presheaves.

(2) Left adjoints send colimits to colimits.

$$\operatorname{colim}_{i \in I} \mathcal{F}_i = (\operatorname{colim}_{PreSh} \mathcal{F}_i)^a.$$

Cor $Sh^{ab}(X_{\acute{e}t})$ is Abelian.

Pf. • Limits exist (b/c can define them pointwise)

- Cokernels exist (b/c colimits exist)

$$\mathrm{coker}(\mathcal{F} \xrightarrow{f} \mathcal{G}) = \mathrm{coeq}(\mathcal{F} \underset{0}{\overset{f}{\rightrightarrows}} \mathcal{G})$$

- im = coim (check on stalks). (exercise)

Next time: injectives, coh.

- start cohomology!