Étale Cohomology

Last time: - proof of Serre's Kähler analogue of RH
- étale morphisms review
- motivation for sites

(1) Sites

Goal: generalize topological spaces/sheaves

(i) need notion of cover

(ii) need notion of intersection (fiber product)

Defn (Grothendieck topology on a category C/site)

The data of, for each $X \in Ob(C)$

a collection of sets of morphisms $\{X_\alpha \to X\}_\alpha$ ← covering families

(0) (intersections exist)

if $X_\alpha \to X$ appears in a covering family,

$Y \to X$ arbitrary

then $X_a \times_X Y$ exist

(1) (Intersecting w/ a cover gives a cover)

If $\{X_\alpha \to X\}$ is a covering family,

$Y \to X$ arbitrary, then

$\{Y \times_X X_\alpha \to Y\}$ is a covering family.

(2) (composition of covers are covers)

If $\{X_\alpha \to X\}_\alpha$, $\{X_{\alpha\beta} \to X_\alpha\}_{\alpha,\beta}$ are covering families, then

$\{X_{\alpha\beta} \to X_\alpha \to X\}$ is a covering family.

(3) (isos are covers) If $X \xrightarrow[\sim]{f} Y$ is an isomorphism, then $\{X \xrightarrow{\sim} Y\}$ is a covering family.

<u>Ex</u> X top space $\quad C = \mathrm{Open}(X)$

$\mathrm{Ob}(C)$ = open subsets of X

Unique morphism $U \to V$ if $U \subseteq V$.

$\{U_\alpha \to U\}$ is a covering family if $\bigcup_\alpha U_\alpha = U$.

<u>Ex</u> X-scheme $\quad \bullet\ X_{ét}$ - category whose objects are étale morphisms $Y \xrightarrow{f} X$

morphisms are maps over X

$$\begin{tikzcd} Y_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & Y_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

$\{U_\alpha \xrightarrow{f_\alpha} U\}$ is a covering family if $\bigcup \mathrm{im}(f_\alpha) = U$.

- $X_{\acute{E}t}$ - category whose objects are all X-schemes
  morphisms are maps $/X$

  $\{U_\alpha \xrightarrow{f_\alpha} U\}$ is a covering family if
  - all f_α are étale
  - $\bigcup \mathrm{im}(f_\alpha) = U$.

Ex X - cpx analytic space

$X_{an\text{-}\acute{e}t}$ - objects are cpx analytic spaces $Y \xrightarrow{f} X$ s.t. locally on Y, f is an analytic iso.

morphisms are morphisms $/X$

covers are covers.

Rem $Sh(X_{an\text{-}\acute{e}t}) \simeq Sh(X^{top})$ (exercise)

Ex fppf topology (faithfully flat + finite presentation)

X_{fppf} - objects are flat, finite presentation $Y \to X$

morphisms are morphisms$/X$

covers are covers.

$$\begin{tikzcd} Y_1 \arrow[rr] \arrow[dr] & & Y_2 \arrow[dl] \\ & X & \end{tikzcd}$$

Ex Nisnevich, Crystalline, infinitesimal site, ..., cdh, arc...

<u>Ex</u> X scheme, $X_{zar} = \mathrm{Open}(X^{top})$
w/ covers as covers.

X_{Zar} - category is all X-schemes
$\{U_\alpha \xrightarrow{f_\alpha} U\}$ is a covering family if
f_α are open embeddings and
$\bigcup \mathrm{im}(f_\alpha) = U$.

<u>Defn</u> (Presheaf on C) A D-valued presheaf is a contravariant functor $F : C \to D$.

<u>Rem</u> Don't need a Groth. top. on C.

<u>Ex</u> X top space, a D-valued presheaf on X is the same as a presheaf on $\mathrm{Open}(X)$.

<u>Defn</u> (Sheaf on a site C)

A sheaf F is a pre-sheaf s.t.

$$F(U) \to \prod_\alpha F(U_\alpha) \overset{F(\pi_1)}{\underset{F(\pi_2)}{\rightrightarrows}} \prod_{\alpha,\alpha'} F(U_\alpha \times_U U_{\alpha'})$$

is an equalizer diagram for all covering families $\{U_\alpha \xrightarrow{f_\alpha} U\}$.

$$\begin{tikzcd}
& U_\alpha \times_U U_{\alpha'} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, phantom, "\square"] & \\
U_\alpha \arrow[dr] & & U_{\alpha'} \arrow[dl, "f_{\alpha'}"] \\
& U &
\end{tikzcd}$$

$f_\alpha \quad V \quad \to_\alpha$

(Suppose F is valued in sets)

(1) $F(U) \to \prod_\alpha F(U_\alpha)$ injective

(the value of F on U is determined by its value on U_α.)

(2) Given $(s_\alpha) \in \prod F(U_\alpha)$ s.t.

$$F(\pi_1)(s_\alpha) = F(\pi_2)(s_\alpha)$$

then (s_α) comes from $F(U)$.

Defn (A morphism of sheaves/pre-sheaves)

A morphism $F_1 \to F_2$ is a natural transformation.

Ex of sheaves on $X_{Ét}$

Thm Any representable functor is a sheaf on $X_{Ét}$.

(Any rep'ble functor is a sheaf on X_{Fppf})

Ex μ_n - functor rep'd by $\mu_n = \operatorname{Spec} k[t]/t^n - 1$

$$\mu_n(U) = \{ f \in \mathcal{O}_U(U) \mid f^n = 1 \}$$

Ex $\mathcal{O}_X^{ét}(U) = \mathcal{O}_U(U)$

rep'ble by $\mathbb{A}^1_X$.

Ex Constant sheaf $\underline{\mathbb{Z}/\ell^n\mathbb{Z}}$.

Rep'd by $(\mathbb{Z}/\ell^n\mathbb{Z}) \times X$

$$\underline{\mathbb{Z}/\ell^n\mathbb{Z}}(U) = \mathrm{Hom}_{cont}(U^{top}, \mathbb{Z}/\ell^n\mathbb{Z})$$

Ex $\mathbb{G}_m(U) = \mathcal{O}_U(U)^\times$

rep'd by $\mathbb{G}_{m,X} = \operatorname{Spec} \mathbb{Z}[t, t^{-1}] \times_{\operatorname{Spec} \mathbb{Z}} X$

Ex $\mathbb{P}^n : U \mapsto \mathrm{Hom}_X(U, \mathbb{P}^n_X)$

Defn (ish) (Étale cohomology w/ coeffs in $\mathbb{Z}/\ell^n\mathbb{Z}$)

I owe you:
- Pf that $\underline{\mathbb{Z}/\ell^n\mathbb{Z}}$ is a sheaf on $X_{ét}$
- Pf that the category of sheaves on $X_{ét}$ w/ values in Ab is Abelian w/ enough injectives

$$\Gamma_X : \mathcal{F} \longmapsto \mathcal{F}(X)$$

$$H^i(X_{ét}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = R^i\Gamma_X(\underline{\mathbb{Z}/\ell^n\mathbb{Z}}).$$

↯ showing cokernels exist in category of Abelian sheaves on a site is non-trivial.
(exercise)

Ex $\mathbb{G}_m \xrightarrow{t^n \leftarrow t} \mathbb{G}_m$ / n invertible on the base

Map of sheaves: X_{zar} : $\mathcal{O}^\times \xrightarrow{f \longmapsto f^n} \mathcal{O}^\times$

$X_{ét}$: $\mathcal{O}^\times_{ét} \xrightarrow{f \longmapsto f^n} \mathcal{O}^\times_{ét}$

__Claim__ This map is in general not an epimorphism on X_{zar}, but it is on $X_{ét}$.

Pf (i) Not an epi on X_{zar}.

$X = \operatorname{Spec} \mathbb{R}$, $n = 2$

Is $\mathbb{R}^\times \xrightarrow{t \mapsto t^2} \mathbb{R}^\times$ surjective? No!

(ii) Is surjective on $X_{ét}$ if n is invertible on X.

Given $f \in \mathbb{G}_m(U)$, want étale cover of U s.t. f obtains an n-th root on that cover?

$$\begin{tikzcd} U \times_{\mathbb{G}_m} \mathbb{G}_m \arrow[r] \arrow[d, "ét"'] & \mathbb{G}_m \arrow[d] & z^n \\ U \arrow[r, "f"] & \mathbb{G}_m & z \arrow[u, mapsto] \end{tikzcd}$$

étale b/c n invertible

__Claim__ f has n-th root upstairs

$\mathbb{A}^1_{U,z} \supseteq V(z^n - f)$ - étale cover of U

z is an n-th root.

(exercise: check details)

<u>Rem</u> $\mathbb{G}_m \xrightarrow{z \mapsto z^2} \mathbb{G}_m$ is an epi in $Sh(X_{fppf})$

<u>Defn</u> (Cts map of sites)

T_1, T_2 sites. A cts map $f: T_1 \to T_2$ is a functor $T_2 \to T_1$ preserves fiber products, sends covering families to covering families.

<u>Ex</u> $f: X \to Y$ cts map of spaces.

$$\begin{aligned} \mathrm{Open}(Y) &\longrightarrow \mathrm{Open}(X) \\ U &\longmapsto f^{-1}(U) \end{aligned}$$

(Exercise: Check this is a cts map of sites)

$\mathrm{End}(E) \supseteq E \qquad \mathrm{Hom}_{\mathrm{gp\,sch}}(E, E)$

$\mathrm{Spec}\, \mathbb{F}_{p^2} \leftarrow$ 1 closed pt

$\mathrm{Spec}\, \mathbb{F}_{p^2} \otimes \overline{\mathbb{F}_p} = \mathrm{Spec}\, \overline{\mathbb{F}_p} \sqcup \mathrm{Spec}\, \overline{\mathbb{F}_p}$

$\# \left| \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^2}, \overline{\mathbb{F}_p}) \right| = 2$