Étale Cohomology - Proper Base Change (cont.)

Thm $\pi : X \to S$ proper, $\tilde{\mathcal{F}} \in Sh^{ab}(X_{et})$ constructible.
Then $R^i\pi_*\tilde{\mathcal{F}}$ are constructible for $i \geq 0$ and

$$(R^i\pi_*\tilde{\mathcal{F}})_{\bar{s}} \simeq H^i(X_{\bar{s},et}, \tilde{\mathcal{F}}|_{X_{\bar{s},et}}) \quad \text{for all geom.}$$

pts $\bar{s} \hookrightarrow S$.

Cor For $\pi : X \to S$ proper, the formation of $R^i\pi_*\tilde{\mathcal{F}}$ ($\tilde{\mathcal{F}}$ torsion) commutes w/ base change.

Key ideas: (1) Reduce to the case where π is a relative curve

(2) Devissage to reduce to the case where $\tilde{\mathcal{F}} = \mu_n$

(3) $\pi : X \to S$ is a relative curve, $\tilde{\mathcal{F}} = \mu_n$

$$0 \to \pi_*\mu_n \to \pi_*\mathbb{G}_m \to \pi_*\mathbb{G}_m \to R^1\pi_*\mu_n \to R^1\pi_*\mathbb{G}_m \to R^1\pi_*\mathbb{G}_m \to R^2\pi_*\mu_n \to 0$$

Goal: $\pi_*\mu_n$, $R^1\pi_*\mu_n$, $R^2\pi_*\mu_n$ are represented by quasi-finite S-schemes.

Key geometric input: (Grothendieck)

In this situation, $R^1\pi_*\mathbb{G}_m = \mathrm{Pic}\, X/S$ is representable by S-scheme, locally of finite type.

$$\mathrm{Pic}_{X/S}(T) = \left(\begin{Bmatrix} \text{line bundles} \\ \text{on } X_T \end{Bmatrix} \Big/ \pi_T^* \begin{Bmatrix} \text{line bundles} \\ \text{on } T \end{Bmatrix} \right)^a$$

$$R^1\pi_*\mu_n = \ker\left(\mathrm{Pic}_{X/S} \xrightarrow{[n]} \mathrm{Pic}_{X/S}\right)$$

quasi-finite

$$R^2\pi_*\mu_n = \mathrm{coker}\left(\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S}\right)$$

quasi-finite.

(n invertible on S)

Ex $\pi : X \to S$ sm. proper curve

$$H^{r+s}(X_{\acute{e}t}, \mathbb{Z}/n\mathbb{Z}) \Leftarrow H^r(S_{\acute{e}t}, R^s\pi_*\mathbb{Z}/n)$$

what is this sheaf.

$R^s\pi_*\mathbb{Z}/n\mathbb{Z}$ is constructible

$(R^s\pi_*\mathbb{Z}/n\mathbb{Z})_{\bar{s}}$ — known, $H^s(X_{\bar{s}}, \mathbb{Z}/n\mathbb{Z})$

order doesn't depend on $\bar{s}$.

Ex $\pi : X \to S$ proper curve $\Rightarrow$ over locus in S where π is smooth, $R\pi_*\mathbb{Z}/n\mathbb{Z}$ locally constant.

"Confusing thing": Locally const sheaf on S, Galois on the fibers "vary".

Prop U separated scheme, $\mathcal{F}$ constructible sheaf on U. Then $H_c(U, \mathcal{F}) := H^i(X_{\acute{e}t}, j_! \mathcal{F})$

$$j : U \xrightarrow{\text{open}} X \leftarrow \text{proper}$$

does not depend on X.

Pf $j_1 : U \hookrightarrow X_1$ $\qquad$ $j_2 : U \hookrightarrow X_2$

Want: $H^j(X_i, j_{i!} \mathcal{F})$ independent of i.

$$\begin{tikzcd} U \arrow[r, "{(j_1, j_2)}"] & X_1 \times X_2 \arrow[d] \arrow[dr] \\ & X_1 & X_2 \end{tikzcd}$$

$X = \overline{\mathrm{im}(j_1, j_2)} \subseteq X_1 \times X_2$

Can assume:

$$\begin{tikzcd} U \arrow[r, hook, "j", "\text{open}"'] \arrow[dr, "j_1"', "\text{open}"] & X \arrow[d, "\pi"] & \text{proper} \\ & X_1 & \text{proper} \end{tikzcd}$$

π proper

$$H^r(X_1, R^s \pi_* j_! \mathcal{F}) \Rightarrow H^r(X, j_! \mathcal{F})$$

$$(R^s \pi_* j_! \mathcal{F})_{\bar{x}} = H^s(\pi^{-1}(\bar{x})_{\acute{e}t}, j_! \mathcal{F})$$

$$= 0 \text{ if } s > 0.$$

$$H^r(X_1, \pi_* j_! \mathcal{F}) = H^r(X, j_! \mathcal{F})$$

(Exercise) $\pi_* j_! \mathcal{F} \simeq j_{1!} \mathcal{F}$. $\boxed{}$

Prop (i) Given $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of constructible abelian sheaves on U_{et}, get LES in $H_c^i(U, -)$.

(ii) $\mathcal{F}$ constructible, $H_c^i(U_{et}, \mathcal{F})$ is finite.

Pf (i) Want LES $H^{i-1}(X_{et}, j_!\mathcal{H}) \to H^i(X_{et}, j_!\mathcal{F}) \to H^i(X_{et}, j_!\mathcal{G})$

$j : U \hookrightarrow X \leftarrow$ proper.

Arises from SES $0 \to j_!\mathcal{F} \to j_!\mathcal{G} \to j_!\mathcal{H} \to 0$

(b/c $j_!$ is exact)

(ii) ETS $j_!\mathcal{F}$ is constructible on X, b/c $H^i(X_{et}, j_!\mathcal{F})$ is finite by proper base change theorem.

Claim $j_!\mathcal{F}$ is constructible

(i) finite stalks: stalks of $\mathcal{F}$ are finite

(ii) $T \subseteq X$ closed, $j_!\mathcal{F}|_T$ is locally constant on an open of T. $\square$

Purity, Gysin sequence, cohomology w/ supports

$\Lambda = \mathbb{Z}/n\mathbb{Z}$ (n invertible on the base)

$Sh^\Lambda \leftarrow$ sheaves of Λ-modules.

Ex μ_n - sheaf of Λ-modules, given $\mathcal{F} \in Sh^{\Lambda}$,

$$\tilde{\mathcal{F}}(r) = \mathcal{F} \otimes_{\Lambda} \mu_n^{\otimes r}.$$

General way of relating coh. on an open to coh. on the complement: cohomology w/ supports.

$$\Gamma_Z : Sh^{ab}(X_{\acute{e}t}) \to Ab \qquad \underset{\text{closed subscheme}}{Z \subseteq X}$$

$$\Gamma_Z(X,-) := \ker(\Gamma(X,-) \to \Gamma(U,-))$$

Exercise Left exact functor

Defn $H^*_Z(X,-)$ - right derived functors of Γ_Z.

Thm Functorial LES $\quad (- \in Sh^{ab}(X_{\acute{e}t}))$

$$H^*_Z(X_{\acute{e}t},-) \to H^*(X_{\acute{e}t},-) \to H^*(U_{\acute{e}t},-) \to H^{*+1}_Z(X_{\acute{e}t},-) \to \cdots$$

Pf $\quad U \underset{\text{open}}{\overset{j}{\hookrightarrow}} X \underset{\text{cl}}{\overset{i}{\hookleftarrow}} Z$

$$0 \to j_! j^* \underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \to i_* i^* \underline{\mathbb{Z}} \to 0$$

Claim $\mathrm{Hom}(i_* i^* \underline{\mathbb{Z}}, \mathcal{F}) \simeq \Gamma_Z(X_{et}, \mathcal{F})$

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}(i_* i^* \underline{\mathbb{Z}}, \mathcal{F}) & \to & \mathrm{Hom}(\underline{\mathbb{Z}}, \mathcal{F}) & \to & \mathrm{Hom}(j_! j^* \underline{\mathbb{Z}}, \mathcal{F}) \\ & \wr\| & & \| & & \| \\ & \Gamma_Z(X_{et}, \mathcal{F}) & & \Gamma(X, \mathcal{F}) & & \mathrm{Hom}(j^* \underline{\mathbb{Z}}, j^* \mathcal{F}) \\ & & & & \searrow & \| \\ & & & & & \Gamma(U, \mathcal{F}|_U) \end{array}$$

$\boxtimes$

$$\Rightarrow H^i_Z(X_{et}, \mathcal{F}) \simeq \mathrm{Ext}^i_{Sh^{ab}(X_{et})}(i_* i^* \underline{\mathbb{Z}}, \mathcal{F})$$

LES we want is just the long exact sequence of Ext^*.

Thm $Z \underset{cl}{\subseteq} X /k\text{-field}$ $\quad Z, X$ smooth, Z is of pure codim'n c in X.

Then for $\mathcal{F} \in Sh^{ab}(X_{et})$ locally constant constructible (lcc)

canonical iso. $$\begin{array}{c} H^{r-2c}(Z, \mathcal{F}(-c)) \\ \downarrow\wr \\ H^r_Z(X, \mathcal{F}). \end{array}$$ For all $r \geq 0$

Ex $Z = pt \subseteq \mathbb{A}^1$ $\quad c = 1 \quad / k = \bar{k}$ of char $\nmid n$

$$H^{r-2}(pt, \mathbb{Z}/n\mathbb{Z}(-1)) \cong H^r_{pt}(\mathbb{A}^1_{\bar{k}}, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}(-1) & \text{if } r = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Pf

$$H^i_{pt}(\mathbb{A}^1, \mathbb{Z}/n\mathbb{Z}) \to H^i(\mathbb{A}^1, \mathbb{Z}/n\mathbb{Z}) \to H^i(\mathbb{G}_m, \mathbb{Z}/n\mathbb{Z}) \to H^{i+1}_{pt}(\mathbb{A}^1, \mathbb{Z}/n\mathbb{Z})$$

$$H^i(\mathbb{A}^1, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$$

$$H^i(\mathbb{A}^1, \mu_n) \to H^i(\mathbb{A}^1, \mathbb{G}_m) \to H^i(\mathbb{A}^1, \mathbb{G}_m) \to H^{i+1}(\mathbb{A}^1, \mathbb{G}_m)$$

$$0 \to \mu_n \to \underset{k^\times}{k[t]^\times} \xrightarrow{[n]} k^\times \to \underset{0}{H^1(\mathbb{A}^1, \mu_n)} \to \underset{0}{\operatorname{Pic}(\mathbb{A}^1)} \to \underset{0}{\operatorname{Pic}(\mathbb{A}^1)} \to 0$$

$$H^i(\mathbb{G}_m, \mu_n) \to H^i(\mathbb{G}_m, \mathbb{G}_m) \to H^i(\mathbb{G}_m, \mathbb{G}_m) \to H^{i+1}(\mathbb{G}_m, \mu_n)$$

$$0 \to \underset{\mu_n(k)}{\mu_n(k)} \to \underset{k^\times \times t^{\mathbb{Z}}}{k[t, t^{-1}]^\times} \to \underset{k^\times \times t^{\mathbb{Z}}}{k[t, t^{-1}]^\times} \to \underset{\mathbb{Z}/n\mathbb{Z}}{H^1(\mathbb{G}_m, \mu_n)} \to \underset{0}{\operatorname{Pic}(\mathbb{G}_m)}$$