Étale Cohomology - Lecture 1

Admin stuff: - In-person
- Please show up synchronously
- Assessment: Some exercises during lecture + reading
  might have to present
- A bit of reading

Prerequisites: • Homological algebra
(abelian categories, derived functors
spectral sequence)
• Sheaf theory, sheaf cohomology
• Schemes (Hartshorne II, III)

Discord: Join w/ full name.

<u>Goals</u>: (1) Basics of étale cohomology
- étale morphism
- Grothendieck topologies, étale topology
- étale coh. — basic thms + comparisons
- Étale coh. of curves.

(Following Milne's "Lectures on Étale Cohomology")

(Frietalg-Kiehl)

(2) Pf(s) of Weil conjectures

(3) Advanced topics: - Weil II

- Formality
- Other topics (monodromy)

(Katz's AWS notes)

What is étale cohomology?

X variety$/\mathbb{C}$

f.t. separated integral $\mathbb{C}$-scheme

$$X(\mathbb{C})^{an} \rightsquigarrow H^i(X(\mathbb{C})^{an}, \mathbb{Z})$$

(i) f.g. $\mathbb{Z}$-modules

(ii) $H^i(X(\mathbb{C})^{an}, \mathbb{C})$ · extra structure

(iii) cycle classes ...

Goal of étale coh: Do a similar thing for general nice schemes.

$$X \text{ "nice scheme"} \rightsquigarrow H^i(X_{\acute{e}t}, \mathbb{Z}/\ell^n\mathbb{Z})$$

$$\rightsquigarrow H^i(X_{\acute{e}t}, \mathbb{Z}_\ell)$$

$\mathbb{Z}_\ell$: ℓ-adic integers

$\rightsquigarrow$

$H^i(X_{ét}, \mathbb{Q}_\ell)$

"twisted coefficients"

e.g. $X = \operatorname{Spec} \mathcal{O}_K$ K : # field

most analogous to $/\mathbb{C}$ $\longrightarrow$ X = variety over alg. closed field

X = variety over non-alg. closed field.

Related invts:

$(X, \bar{x}) \rightsquigarrow \pi_1^{ét}(X, \bar{x})$

$\bar{x}$ geom. pt; $\pi_1^{ét}(X, \bar{x})$ profinite grp

More invts (beyond scope of this course)

- higher htpy grps
- htpy type

Note: This cohomology theory is weird!

<u>Thm</u> (Serre) There DNE a cohomology theory for schemes$/\overline{\mathbb{F}}_q$ with the following properties:

(1) Functorial

(2) Kunneth

(3) E ell. curve $\rightsquigarrow H^1(E) = \mathbb{Q}^2$.

Slogan: There's no cohomology theory w/ $\mathbb{Q}$ coefficients.

Pf Take E - s.s. elliptic curve.

$$\mathrm{End}(E) \otimes \mathbb{Q} = \text{quaternion algebra. } R \quad \text{(non-split.)}$$

Fact There are no $R \to \mathrm{Mat}_{2\times 2}(\mathbb{Q})$.

Exercise: Functoriality + Künneth $\Rightarrow$ $\mathrm{End}(E) \curvearrowright E$

$$\downarrow$$

$$\mathrm{End}(E) \curvearrowright H^1(E)$$

$$\downarrow$$

$$\mathrm{End}(E) \to \mathrm{Mat}_{2\times 2}(\mathbb{Q}) \quad \} \ \boxtimes$$

Exercise Prove the same thing for $\mathbb{Q}_p$ coefficients where $p \mid \mathrm{char}(k)$.

Weil Conjectures:

X - variety / $\mathbb{F}_q$ (geom. integral f.t. separated $\mathbb{F}_q$-scheme)

$$S_X(t) = \exp\left(\sum_{n>0} \frac{\# X(\mathbb{F}_{q^n})}{n} t^n \right)$$

- $\frac{\partial}{\partial t} \log S_X(t)$ = gen ftn for $\# X(\mathbb{F}_{q^n})$

slogan: locations of zeros + poles of a meromorphic ftn control the growth rate of the coefficients of the Taylor series of its log-derivative.

Exercise: make slogan precise for rat'l ftns

Conj (Weil) (1) $\zeta_X(t)$ is a rat'l ftn

(ftnl eqn) (2) X sm. proper of dim n

$$\text{Then } \zeta_X(q^{-n}t^{-1}) = \pm q^{\frac{nE}{2}} t^E \zeta_X(t)$$

E: "Euler char" ???

(RH) (3) All roots + poles of $\zeta_X(t)$ have abs value $q^{i/2}$ $i \in \mathbb{Z}$.

(4) X-sm. proper. # of roots/poles w/ abs value $q^{-i/2}$ = i-th "Betti number" of $X_{\overline{\mathbb{F}_q}}$.

makes sense if X lifts to char 0.

(make sense of this in general)

Pf (1) Dwork (will follow from f.d. of H^i)

(2) Grothendieck (will follow from Poincaré duality)

(3.4) Deligne

<u>Euler product</u>

$$\zeta_X(t) = \exp\Big(\sum_{n>0} \frac{\# X(\mathbb{F}_{q^n})}{n} t^n \Big)$$

(exercise)

$$= \prod_{x \in |X|} \exp\Big(t^{\deg(x)} + \frac{t^{2\deg(x)}}{2} + \cdots \Big)$$

closed pts of X → $|X|$

$$= \prod_{x \in |X|} \exp\big(-\log(1 - t^{\deg(x)})\big)$$

$$= \prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}} \quad \longleftarrow \text{Euler product}$$

$$= \prod_{x \in |X|} \big(1 + t^{\deg(x)} + t^{2\deg(x)} + \cdots \big)$$

$$= \sum_{n \geq 0} \left(\begin{matrix} \#\text{ of Galois-stable subsets of} \\ X(\overline{\mathbb{F}}_q) \text{ of size } n \end{matrix} \right) t^n$$

$$= \sum_{n \geq 0} \#[\mathrm{Sym}^n(X)(\mathbb{F}_q)]\, t^n$$

$\mathrm{Sym}^n(X) = X^n / \Sigma_n$

Thm X sm. proper curve $/\mathbb{F}_q$.

$\zeta_X(t)$ is rat'l.

Pf $\mathrm{Sym}^n X \to \mathrm{Pic}^n X$

$D \longmapsto \mathcal{O}(D)$

Fiber $/\mathcal{O}(D) = \mathbb{P}\Gamma(X, \mathcal{O}(D))$

$\dim \mathbb{P}\Gamma(X, \mathcal{O}(D)) = \deg(D) - g + \dim H^1(X, \mathcal{O}(D))$

If $\deg D > 2g-2$, then $H^1(X, \mathcal{O}(D)) = 0$

$$H^1(X,\mathcal{O}(D)) = H^0(X, \mathcal{O}(K-D))^\vee = 0.$$

negative degree

- Fibers of $\mathrm{Sym}^n X \to \mathrm{Pic}^n X$ are $\simeq \mathbb{P}^{n-g}$ if $n > 2g-2$
- WLOG (exercise) assume $X(\mathbb{F}_q) \neq \emptyset$.

$\Rightarrow \mathrm{Pic}^n(X) \simeq \mathrm{Pic}^{n+1}(X)$ for all n.

$\# \mathrm{Sym}^n(X)(\mathbb{F}_q) = \# \mathbb{P}^{n-g}(\mathbb{F}_q) \cdot \# \mathrm{Pic}^0(X)(\mathbb{F}_q)$

for all $n > 2g-2$.

- $\zeta_X(t) =$ polynomial $+ \sum_{n > 2g-2} t^n \cdot \# \mathrm{Pic}^0(X)(\mathbb{F}_q) \cdot (1+q+q^2+\cdots+q^{n-g})$

<u>Exercise</u> Show this is a rat'l ftn.

<u>Thm</u> (Functional Eqn)

$$\zeta_X(q^{-1}t^{-1}) = \pm q^{\frac{(2-2g)}{2}} t^{2-2g} \zeta_X(t)$$

<u>Pf</u> (Exercise) Serre duality.

RH: later.

<u>Thm</u> (Dwork) $X/\mathbb{F}_q$ variety. Then

$\zeta_X(t) =$ rat'l ftn.

<u>Pf</u> (Grothendieck) Idea: Frob: $X \to X$

$X(\mathbb{F}_q) =$ fixed pts of Frob on $X_{\overline{\mathbb{F}}_q}$.

Use "Lefschetz fixed pt formula":

$$\# X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(\mathrm{Frob}^n \mid H^i_c(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell))$$

$\ell \neq \operatorname{char} \mathbb{F}_q$.

<u>Lemma</u> V f.d. vector space, $F: V \to V$. Then

$$\exp\left(\sum \frac{\operatorname{Tr}(F^n)}{n} t^n\right) \text{ is rat'l.}$$

<u>Pf</u> Suffices to treat the case $\dim V = 1$

$$\exp\left(\sum \frac{\alpha^n}{n} t^n\right) = \exp(-\log(1-\alpha t))$$
$$= \frac{1}{1-\alpha t} \leftarrow \text{rat'l} \quad \square.$$

Key input: · Lefschetz fixed pt formula
· f.d. of étale cohomology.

Exercise: Try to figure out how Poincaré duality should give fnl eqns

Thm (Serre) X sm. proj. variety / $\mathbb{C}$

$[H] \in H^2(X(\mathbb{C})^{an}, \mathbb{Z})$ is a hyperplane class.

$F : X \to X$ s.t. $F^*[H] = q[H]$, $q \in \mathbb{Z}_{\geq 1}$

$$L(F^n) = \sum_{i=0}^{2\dim X} (-1)^i \, \mathrm{Tr}\big((F^*)^n \,|\, H^i(X, \mathbb{Q})\big)$$

$$S_{X,F}(t) = \exp\Big(\sum_{n=1}^{\infty} \frac{L(F^n)}{n} t^n\Big)$$

Then $S_{X,F}(t)$ satisfies RH (equiv: eigenvalues of F^* acting on $H^i(X, \mathbb{C})$ all have abs. value $q^{i/2}$.

Next time: Prove this + étale morphisms.
(review def'n of étale morphism)