# Étale cohomology – 9/24/2020

Last time: Leray spectral sequence

$$\begin{array}{c} X \\ \downarrow f \\ Y \\ \downarrow g \\ Z \end{array} \rightsquigarrow R^i g_* R^j f_* (-) \Rightarrow R^{i+j}(g \circ f)_*(-)$$

<u>Ex</u> $\begin{array}{c} X \\ \downarrow \pi \\ \operatorname{Spec} k \end{array}$ $k$ - not alg. closed

Leray s.s. $H^i(k, R^j \pi_* \mathcal{F}) \Rightarrow H^{i+j}(X_{ét}, \mathcal{F})$

Galois coh.

The Galois module corresponding to $R^j \pi_* \mathcal{F} = H^j(X_{k^s}, \mathcal{F})$

$$\rightsquigarrow H^i(k, H^j(X_{k^s, ét}, \mathcal{F})) \Rightarrow H^{i+j}(X_{ét}, \mathcal{F})$$

<u>Sub-ex</u> $k$ - finite field, $X/k$ sm. proj. variety

$$H^i(k, V) = \begin{cases} V^G & i=0 \\ V_G & i=1 \end{cases} \qquad 0 \quad i>1$$

<u>Rem</u> $\begin{array}{c} X \\ \pi \downarrow \\ Y \end{array}$ sm. proper morphism of varieties$/\mathbb{C}$

$$H^i(Y, R^j \pi_* \underline{\mathbb{Q}}) \Rightarrow H^{i+j}(X, \underline{\mathbb{Q}})$$

<u>Fact</u> (Deligne) This s.s. degenerates at $E_2$.

<u>Prop</u> $\begin{array}{c} X \\ \pi \downarrow \\ Y \end{array}$ $R^i \pi_* \mathcal{F}$ = sheaf associated to presheaf

$$U \longmapsto H^i(\pi^{-1}(U)_{ét}, \mathcal{F})$$

**Pf** $\mathcal{F} \to \mathcal{I}^\bullet$ injective res'n

$$\mathcal{H}^i(\pi_* \mathcal{I}^\bullet) =: R^i \pi_* \mathcal{F}$$

exact

$$\begin{tikzcd} \mathrm{PreSh}(X_{\acute{e}t}) \arrow[r, "\pi_*"] & \mathrm{PreSh}(Y_{\acute{e}t}) \arrow[d, "a"] \\ \mathrm{Sh}(X_{\acute{e}t}) \arrow[u, "\mathrm{Forget}"] \arrow[r, "\pi_*"] & \mathrm{Sh}(Y_{\acute{e}t}) \end{tikzcd}$$

sheafification

$$\begin{aligned} R^i \pi_* \mathcal{F} &= \mathcal{H}^i(\pi_* \mathcal{I}^\bullet) \\ &= \mathcal{H}^i(a \circ \pi_* \circ \mathrm{Forget}(\mathcal{I}^\bullet)) \\ &= a \circ \pi_* \left( \mathcal{H}^i(\mathrm{Forget}(\mathcal{I}^\bullet)) \right) \end{aligned}$$

$$\left( U \mapsto H^i(U, \mathcal{F}) \right)$$

$$\overbrace{(U \xrightarrow{\acute{e}t} Y) \mapsto H^i(\pi^{-1}(U), \mathcal{F})}$$

$\boxed{}$

**Ex** $X$ integral scheme, $\eta \xrightarrow{\iota} X$ generic pt.

$\mathcal{F} \in \mathrm{Sh}(\eta_{\acute{e}t})$. Goal: Understand $R^i \iota_* \mathcal{F}$

$\bar{x} \to X$ geom. pt: compute $(R^i \iota_* \mathcal{F})_{\bar{x}}$

$$\begin{aligned} (R^i \iota_* \mathcal{F})_{\bar{x}} &= \varinjlim_{(U, \bar{u})} (R^i \iota_* \mathcal{F})(U) \\ &= \varinjlim_{(U, \bar{u})} H^i(U_\eta, \mathcal{F}|_{U_\eta}) \end{aligned}$$

**Claim** (exercise) $\mathcal{O}_{X, \bar{x}}$ = stalk of $\mathcal{O}_X$ at $\bar{x}$, ← strictly Henselian local ring of $X$ at $\bar{x}$

$K_{\bar{x}} := \mathrm{Frac}(\mathcal{O}_{X, \bar{x}})$. ← strictly Henselian field associated to $X$ at $\bar{x}$.

$$(R^i \iota_* \mathcal{F})_{\bar{x}} = H^i(K_{\bar{x}}, \mathcal{F}|_{K_{\bar{x}}})$$

**Goal** Understand $H^i(X, \mathbb{G}_m)$ $(X$ - curve$/k = k^s)$.
for $i > 1$.

Want to reduce this question to questions in Galois cohomology.

**Prop** $X$ regular variety$/k$. $\eta \hookrightarrow X$ is gen. pt.
Then there is a SES in $Sh(X_{ét})$.

$$0 \to \mathbb{G}_m \xrightarrow{res} \eta_* \mathbb{G}_m \xrightarrow{div} \bigoplus_{\substack{z \in X \text{ codim} \\ 1}} \iota_{z*} \mathbb{Z} \to 0$$

**Pf** (of exactness)

(1) $\mathbb{G}_m \to \eta_* \mathbb{G}_m$ injective $\qquad \mathbb{G}_m(U) \to \mathbb{G}_m(U_\eta)$ injective

$\mathcal{O}_U^\times \to \bigoplus \mathcal{O}_{\eta_i}$

$\eta_i$ - gen pt of $U$.

(2) $f \in \eta_* \mathbb{G}_m(U)$ s.t. $div(f) = 0 \Rightarrow$

$f$ comes from $\mathbb{G}_m(U)$.

ETS: $f$ is regular (b/c then $f^{-1}$ regular)

$$A = \bigcap_{p \text{ ht } 1} A_p \quad \leftarrow \text{commutative alg. fact.} \quad (\text{by normality})$$

regularity $\Downarrow$ normality

(3) $\eta_* \mathbb{G}_m \to \bigoplus_{z \text{ codim } 1} \iota_{z*} \mathbb{Z}$ is surjective.

Every Weil divisor is loc. principal, i.e. Cartier. True by regularity. ∎

**Cor** LES

compute w/ Leray s.s.

$$\cdots \to H^{i-1}(X_{ét}, \bigoplus_{z \text{ codim } 1} \iota_{z*} \mathbb{Z}) \to H^i(X_{ét}, \mathbb{G}_m) \to H^i(X_{ét}, \eta_* \mathbb{G}_m) \to H^i(X_{ét}, \bigoplus \iota_{z*} \mathbb{Z})$$

$\subseteq \ldots$

**Prop** $X$ curve$/k = k_s$.

$$H^i(X_{\acute{e}t}, \bigoplus_{z \in X \text{ cod } 1} \iota_{z*}\mathbb{Z}) = 0 \quad \text{for } i > 0.$$

**Pf** ETS for $z \in X$ cod 1, $H^i(X_{\acute{e}t}, \iota_{z*}\mathbb{Z}) = 0$ for $i > 0$.

Leray s.s. $H^i(X_{\acute{e}t}, R^j\iota_{z*}\mathbb{Z}) \Rightarrow H^{i+j}(z_{\acute{e}t}, \mathbb{Z})$

$$R^j\iota_{z*}\mathbb{Z} = \begin{cases} \iota_{z*}\mathbb{Z} & j = 0 \\ 0 & j > 0 \end{cases} \Rightarrow \text{s.s. degenerate}$$

$$H^s(z_{\acute{e}t}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & s = 0 \\ 0 & s > 0 \end{cases}$$

$$H^i(X_{\acute{e}t}, \iota_{z*}\mathbb{Z}) = H^i(z_{\acute{e}t}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases} \quad \boxtimes$$

**Cor** $X$ sm. curve$/k = k_s$

$$H^i(X_{\acute{e}t}, \mathbb{G}_m) \xrightarrow{\sim} H^i(X_{\acute{e}t}, \eta_*\mathbb{G}_m) \quad \text{for } i > 1.$$

compute this sp. (new goal)

Leray s.s. $H^i(X_{\acute{e}t}, R^j\eta_*\mathbb{G}_m) \Rightarrow H^{i+j}(\eta, \mathbb{G}_m)$

$$(R^j\eta_*\mathbb{G}_m)_{\bar{x}} \simeq H^j(K_{\bar{x}}, \mathbb{G}_m)$$

<u>Goal</u> <u>Thm</u> $K$ = function field of a curve/alg. closed field
or $K = K_{\bar{x}}$ - strictly Henselian field associated
to a geom. pt of a curve/alg. closed
field.

Then $H^i(K, \mathbb{G}_m) = 0$ for $i > 0$.

This suffices b/c (1) $R^j\eta_* \mathbb{G}_m = 0$ for $j > 0$.

$$H^i(X_{\acute{e}t}, \eta_* \mathbb{G}_m) \simeq H^i(\eta, \mathbb{G}_m) = 0 \text{ if } i > 0$$

Upshot: Reduced computation of étale coh. of a curve to
Galois cohomology.

<u>Brauer gps</u>

$X$ - scheme

<u>Defn</u> (Cohomological Brauer gp)

$$Br^{coh}(X) = H^2(X_{\acute{e}t}, \mathbb{G}_m)_{tors}$$

Understand this geometrically in terms of $PGL_n$-torsors.

<u>Claim</u> $\bigcup_n \left\{ \begin{matrix} \text{étale-locally split} \\ PGL_n\text{-torsors} \end{matrix} \right\} \to H^2(X_{\acute{e}t}, \mathbb{G}_m)$

Idea: $1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$ (SES in $Sh^{gps}(X_{\acute{e}t})$)

(Right exact: $GL_n \to PGL_n$ is smooth, hence has sections
Zariski locally).

Use LES $\cdots \to H^1(X_{\acute{e}t}, PGL_n) \to H^2(X_{\acute{e}t}, \mathbb{G}_m) \to \cdots$

$T$ - étale-locally trivial $PGL_n$-torsor

$\rightsquigarrow [T] \in H^2(X_{ét}, \mathbb{G}_m)$

$[T] \in \check{H}^1(X_{ét}, PGL_n)$

Choose a trivializing $\mathcal{U} \to X$

s.t. $T|_{\mathcal{U}} = PGL_n|_{\mathcal{U}}$

&

cocycle in $PGL_n(\mathcal{U} \times_X \mathcal{U})$ (next time!)

$\mathcal{O}^{\oplus n} \to \mathcal{L}^{\oplus n}$

$\downarrow$

end. of a trivial proj. bundle