Étale cohomology – 9/22/2020

Rem What I've been calling a torsor — some sources call a pseudotorsor
- locally trivial torsor — " torsor

Thm (Hilbert 90)

$$H^1(X_{Zar}, GL_n) \xleftarrow{\sim} H^1(X_{et}, GL_n) \xleftarrow{\sim} H^1(X_{fppf}, GL_n)$$

Pf

$\tau = X_{Zar}, X_{et}, X_{fppf}$

Observe: The data of a GL_n-torsor split by some τ-cover $U \to X$ is the same as descent data for a vector bundle relative U/X.

$$\begin{tikzcd} U \times_X U \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\ U \arrow[d, "\pi"] \\ X \end{tikzcd}$$

$\pi^* T \simeq \pi^* G$ as a $\pi^* G$-torsor

$$\begin{tikzcd} \pi_1^* \pi^* T \arrow[r, "\sim"] \arrow[d, "\wr"] & \pi_2^* \pi^* T \arrow[d, "\wr"] \\ \pi_1^* \pi^* G \arrow[r] & \pi_2^* \pi^* G \simeq G|_{U \times_X U} \end{tikzcd}$$

section to G

FPPF descent $\Rightarrow$ descent data for v.b.'s is effective.

$\Rightarrow H^1(X_\tau, GL_n) \simeq$ n-dim'l v.b.'s on X. ☐

(Exercise) Find other gps for which Hilbert 90 true/not true.

Rem • G - affine, flat X-gp scheme

Q Are all G-torsors rep'ble by a X-scheme

A Yes (by same proof as last time)

Q Given a G-torsor T, fppf-locally trivial — is étale locally trivial.

A No in general, yes if G is smooth.

Pf sketch:

$$\begin{tikzcd} & T \times_X T \arrow[r] \arrow[d] & T \arrow[d] \\ & T \arrow[r, "\text{smooth}"'] \arrow[u, bend right, "s"'] & X \\ U \arrow[ur, "cl"] \arrow[urr, "\text{étale cover}"'] & & \end{tikzcd}$$

trivial

☐

Thm (Hilbert 90)

$$H^1(X_{Zar}, GL_n) \xleftarrow{\sim} H^1(X_{ét}, GL_n) \xleftarrow{\sim} H^1(X_{fppf}, GL_n)$$

Ex $X = \mathrm{Spec}\, k$, $n=1$

$$H^1((\mathrm{Spec}\, k)_{Zar}, \mathbb{G}_m) = 0 = H^1((\mathrm{Spec}\, k)_{ét}, \mathbb{G}_m) = H^1(\mathrm{Gal}(k^s/k), \overline{k}^*)$$

Ex X any scheme, $n=1$

$$H^1(X_{ét}, \mathbb{G}_m) = \mathrm{Pic}(X)$$

Ex $H^1(X_{ét}, \mu_\ell)$ (ℓ invertible on X)

Use: $1 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{z \mapsto z^\ell} \mathbb{G}_m \to 1$ (Kummer sequence)

$$\overset{LES}{\rightsquigarrow} 0 \to H^0(X_{ét}, \mu_\ell) \to \underset{k^*}{H^0(X_{ét}, \mathbb{G}_m)} \xrightarrow{z \mapsto z^\ell} \underset{k^*}{H^0(X_{ét}, \mathbb{G}_m)} \to H^1(X_{ét}, \mu_\ell) \to \mathrm{Pic}(X) \xrightarrow{[\ell]} \mathrm{Pic}(X) \to H^2(X_{ét}, \mu_\ell) \to \cdots$$

Suppose $H^0(X, \mathcal{O}_X) = k = \overline{k}$.

$$H^0(X_{ét}, \mu_\ell) = \mu_\ell(k)$$

$$H^1(X_{ét}, \mu_\ell) = \mathrm{Pic}(X)[\ell]$$

Ex $H^1(X_{ét}, \mathbb{Z}/\ell\mathbb{Z})$ $X/k = \overline{k}$ ℓ invertible in k

Claim $\underset{\mathrm{Hom}(-,\, \mathrm{Spec}\, k[t]/t(t-1)\cdots(t-\ell+1))}{\underline{\mathbb{Z}/\ell\mathbb{Z}}} \simeq \overset{\mathrm{Spec}\, k[t]/t^\ell - 1}{\mu_\ell}$ (depends on a choice of primitive ℓ-th root of unity)

Cor If $\mu_\ell \subseteq k$, $H^1(X_{ét}, \underline{\mathbb{Z}/\ell\mathbb{Z}}) \simeq H^1(X_{ét}, \mu_\ell)$

not Galois equivariant

Geometric interpretation

X affine scheme: $H^1(X_{\acute{e}t}, \mathbb{F}_p) = \mathrm{coker}(\mathcal{O}_X \xrightarrow{x^p - x} \mathcal{O}_X)$ $p = \mathrm{char}\, k$

$k = \bar{k}$

$H^1(X_{\acute{e}t}, \mathbb{Z}/\ell\mathbb{Z})$ = term in LES

$\mathbb{F}_p$-torsors $\mathbb{Z}/\ell\mathbb{Z}$-torsors

Q How does one explicitly write down these torsors?

$$[Y] \in H^1(X_{\acute{e}t}, \mathbb{F}_p) = \mathrm{coker}(\mathcal{O}_X \xrightarrow{x^p - x} \mathcal{O}_X)$$

$(p = \mathrm{char}\, k)$ $Y = \{y^p - y = a\}$ $a \in \mathcal{O}_X$

(Artin-Schreier)

Q $\ell \neq \mathrm{char}\, k$, $\mathrm{Pic}(X) = 0$ $[Z] \in H^1(X_{\acute{e}t}, \mu_\ell) = \mathrm{coker}(\mathcal{O}_X^\times \xrightarrow{x \mapsto x^\ell} \mathcal{O}_X^\times)$

$Z = \{z^\ell = f\}$ $f \in \mathcal{O}_X^\times$

Rem Explicit geometric class field theory — recipe for writing down Abelian covers of curves.

Cohomology of Curves

Goal: Thm X sm. curve $/k = \bar{k}$. Then

$$H^i(X_{\acute{e}t}, \mathbb{G}_m) = \begin{cases} \mathcal{O}_X(X)^\times & i = 0 \quad ✓ \\ \mathrm{Pic}(X) & i = 1 \quad ✓ \\ 0 & i > 1 \quad ? \end{cases}$$

Cor X sm. proper conn'd curve of genus g $/k=\bar{k}$, $\ell \neq \operatorname{char} k$.

$$H^i(X_{et}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \begin{cases} \mathbb{Z}/\ell^n\mathbb{Z} & i=0 \\ \operatorname{Pic}(X)[\ell^n] = (\mathbb{Z}/\ell^n\mathbb{Z})^{2g} & i=1 \\ \mathbb{Z}/\ell^n\mathbb{Z} & i=2 \\ 0 & i>2 \end{cases}$$

Pf of Cor

$$0 \to \operatorname{Jac}(X) \to \operatorname{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

$\operatorname{Jac}(X) = \operatorname{Pic}^0(X)$

Black box: $\operatorname{Jac}(X)$ is g-dim'l AV

$\operatorname{Jac}(X)[\ell^n] \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$

$\operatorname{Jac}(X)(k)$ = divisible

Kummer sequence: $1 \to \underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\mu_{\ell^n}} \to \mathbb{G}_m \to \mathbb{G}_m \to 1$

$$\rightsquigarrow 0 \to H^1(X_{et}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) \to \operatorname{Pic}(X) \xrightarrow{[\ell^n]} \operatorname{Pic}(X) \to H^2(X_{et}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) \to 0$$

$H^1(X_{et}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \operatorname{Pic}(X)[\ell^n] = \operatorname{Jac}(X)[\ell^n] = (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$

$H^2(X_{et}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \operatorname{coker}(\operatorname{Pic}(X) \xrightarrow{[\ell^n]} \operatorname{Pic}(X)) = \operatorname{coker}(\mathbb{Z} \xrightarrow{[\ell^n]} \mathbb{Z}) = \mathbb{Z}/\ell^n\mathbb{Z}$

Exercise Check this using black box + snake lemma.

Rem Isom's **not** Galois equivariant.

Goal: $H^i(X_{\acute{e}t}, \mathbb{G}_m) = 0$ for $i > 1$
(X sm. curve / alg. closed field)

Three ingredients: (1) Leray spectral sequence
(2) Divisor exact sequence
(3) Brauer groups

Pushforwards + Leray s.s.

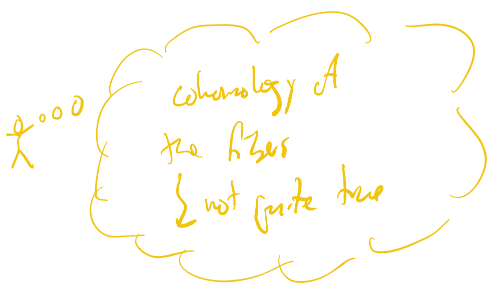
$f : X \to Y$ morphism of schemes

$\rightsquigarrow$

$f_* : Sh(X_{\acute{e}t}) \to Sh(Y_{\acute{e}t})$

$\mathcal{F} \mapsto f_*\mathcal{F}(U \xrightarrow[\acute{e}t]{\pi} Y) = \mathcal{F}(U \times_Y X)$

left-exact functor $\rightsquigarrow R^i f_* : Sh^{ab}(X_{\acute{e}t}) \to Sh^{ab}(Y_{\acute{e}t})$



$R^i f_* \mathcal{F}$ = sheafification of presheaf $V \longmapsto H^i(f^{-1}(V), \mathcal{F})$

Prop f finite morphism (e.g. closed immersion)
then $R^i f_* = 0$ for $i > 0$.

Pf Claim f_* is (right) exact.

must do

Pf Check on stalks (exercise)

$\hookrightarrow$ stalk of $f_* \mathcal{F}$ at $\bar{y} \in Y$ is just $\bigoplus_{\bar{x} \in f^{-1}(\bar{y})} \mathcal{F}_{\bar{x}}$.

for f a finite morphism.

Prop f_* preserves injectives.

Pf True for any functor w/ an exact left adjoint (f^*) (exercise)

Cor (Leray s.s.)

$f: X \to Y$, $g: Y \to Z$

s.s.: $R^i g_* R^j f_* \mathcal{F} \Rightarrow R^{i+j}(g \circ f)_* \mathcal{F}$

Special case: $Z = \operatorname{Spec} k$, $k = \bar{k}$

$$H^i(Y, R^j f_* \mathcal{F}) \Rightarrow H^{i+j}(X, \mathcal{F})$$

Pf Spectral sequence of a composition of functors (Tohoku)

Explicit: $R^i f_* \mathcal{F} = \mathcal{H}^i(f_* \mathcal{J}^\bullet)$ $\mathcal{F} \to \mathcal{J}^\bullet$ injective res'n

$f_* \mathcal{J}^\bullet \leftarrow$ complex of injectives.

Want: $\mathcal{H}^{i+j}(g_* f_* \mathcal{J}^\bullet) = R^{i+j}(g \circ f)_* \mathcal{F}$

Take s.s. of filtered complex $f_* I^\bullet$

filtration: $\tau_{\leq p} f_* I^\bullet$ $\quad R^{p+1} f_* \mathcal{F}$

Ex (differential) $0 \to \tau_{\leq p} f_* I^\bullet \to \tau_{\leq p+1} f_* I^\bullet \to \mathcal{H}^{p+1}(f_* I^\bullet) \to 0$

$\downarrow Rg_*$

LES: $\to R^q g_* R^{p+1} f_* \mathcal{F} \xrightarrow{\delta} R^{q+1} g_* \tau_{\leq p} f_* I^\bullet$

differential on E_2-page of s.s.