Étale Cohomology 10/29/2020

Last time: Cup products + Künneth formula.

Cup products: in Čech cohomology X, Y k-schemes
$U \to X$, $V \to Y$ $\quad$ $\mathcal{F}, \mathcal{G}$ sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on X_{et}, Y_{et}, resp.

$$H^i(\check{C}(U/X, \mathcal{F})) \otimes H^j(\check{C}(V/Y, \mathcal{G}))$$
$$\downarrow$$
$$H^{i+j}(\check{C}(U \times V/X \times Y, \mathcal{F} \boxtimes \mathcal{G}))$$

$$\mathrm{Tot}(\check{C}(U/X, \mathcal{F}) \otimes \check{C}(V/Y, \mathcal{G})) \to \check{C}(U \times V/X \times Y, \mathcal{F} \boxtimes \mathcal{G})$$

$$\begin{array}{ccc} & X \times Y & \\ {}^{\pi_1}\swarrow & & \searrow{}^{\pi_2} \\ X & & Y \end{array} \qquad \mathcal{F} \boxtimes \mathcal{G} = \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G}$$

$f \in \check{C}^i(U/X, \mathcal{F})$, $g \in \check{C}^j(V/Y, \mathcal{G})$

$$f, g \longmapsto f \boxtimes g \in \check{C}^{i+j}(U \times V/X \times Y, \mathcal{F} \boxtimes \mathcal{G})$$

<u>Rem</u> Cup products exist in derived functor sheaf

cohomology (Iversen's book)

Concrete form of Künneth:

(1) X, Y proper varieties$/k$, $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$,
$\Lambda = \mathbb{Z}_\ell$

If $H^*(X,\Lambda)$ or $H^*(Y,\Lambda)$ are free Λ-modules,
then
$$\cup : H^*(X,\Lambda) \otimes H^*(Y,\Lambda) \to H^*(X \times Y, \Lambda)$$
is an isom of graded gps (rings)

Ring structure:
$$H^*(X,\Lambda) \otimes H^*(X,\Lambda) \to H^*(X \times X, \Lambda) \xrightarrow{\Delta^*} H^*(X,\Lambda).$$

(2) X, Y as above (proper)
$$\cup : H^*(X, \mathbb{Q}_\ell) \otimes H^*(Y, \mathbb{Q}_\ell) \to H^*(X \times Y, \mathbb{Q}_\ell)$$
is an isom. of graded rings.

Fancy version of Künneth:

X, Y proper k-schemes, $\mathcal{F} \in Sh(X_{\acute{e}t})$ constr.
$k = \bar{k}$. $\mathcal{G} \in Sh(Y_{\acute{e}t})$ constr.

Then $R\Gamma(X_{et}, \mathcal{F}) \overset{L}{\otimes} R\Gamma(Y_{et}, \mathcal{G})$

$\downarrow \cup$

$R\Gamma((X \times Y)_{et}, \mathcal{F} \overset{L}{\boxtimes} \mathcal{G})$

in $D^b(Ab)$ is a quasi-iso.

<u>Ex</u> $H^*(C \times \mathbb{P}^1, \mathbb{Q}_\ell)$ $\qquad C$ = sm. proper curve $/k = \bar{k}$

$$H^*(C, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & * = 0 \\ V_\ell(\mathrm{Jac}\, C)(-1) & * = 1 \\ \mathbb{Q}_\ell(-1) & * = 2 \end{cases}$$

$V_\ell(\mathrm{Jac}\, C) = T_\ell(\mathrm{Jac}\, C) \otimes \mathbb{Q}_\ell$

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$$H^*(C \times \mathbb{P}^1, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & * = 0 \\ V_\ell(\mathrm{Jac}\, C)(-1) & * = 1 \\ \mathbb{Q}_\ell(-1) \oplus \mathbb{Q}_\ell(-1) & * = 2 \\ V_\ell(\mathrm{Jac}\, C)(-2) & * = 3 \\ \mathbb{Q}_\ell(-2) & * = 4. \end{cases}$$

What info does $H^*(X_{\bar{k}, et}, \mathbb{Q}_\ell)$ carry?

(1) $/\mathbb{F}_q$: point counts

(2) $/k$ = # field Galois reps on $H^*(X_{\bar{k}, et}, \mathbb{Q}_\ell)$

conjecturally carry geometric info.

Pf Sketch (1) Projection formula for $\mathbb{Z}_\ell$-sheaves

Claim $f : X \to S$, $\mathcal{F}$ is a flat $\mathbb{Z}_\ell$-sheaf (or $\mathbb{Z}/\ell^n$) on X_{et}

$\mathcal{G}^\bullet$ bdd above complex of Abelian sheaves on S.

$$(Rf_*\mathcal{F}) \otimes \mathcal{G} \longrightarrow Rf_*(\mathcal{F} \otimes f^*\mathcal{G})$$

is a q.i.

Idea: If $\mathcal{G} = \mathbb{Z}_\ell$, identity map.

reduce to this case.

Ex $H^i(X_{\bar{k}}, \mu_\ell) = H^i(X_{\bar{k}}, \mathbb{Z}/\ell) \otimes \mu_\ell$.

Case $\mathcal{F} = \mathbb{Z}/\ell$, $\mathcal{G} = \mu_\ell$, $f : X_{\bar{k}} \to \operatorname{Spec} \bar{k}$.

(2) $X \times Y$

$$
\begin{tikzcd}
 & {} \arrow[dl, "p"'] \arrow[d, "h"] \arrow[dr, "q"] & \\
\mathcal{F} \curvearrowright X \arrow[r, "f"'] & \operatorname{Spec}\bar{k} = S & Y \curvearrowleft \mathcal{G} \arrow[l, "g"]
\end{tikzcd}
$$

$\mathcal{F}, \mathcal{G}$ constructible sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules

Assume: $\mathcal{F}$ sheaf of flat $\mathbb{Z}/\ell^n\mathbb{Z}$-modules

$$
\begin{aligned}
& Rf_*\mathcal{F} \otimes Rg_*\mathcal{G} \\
& \quad \downarrow \wr \qquad \text{proj. formula} \\
& Rf_*(\mathcal{F} \otimes f^* Rg_*\mathcal{G}) \\
& \quad \downarrow \wr \qquad \text{(proper base change)} \\
& Rf_*(\mathcal{F} \otimes Rp_* q^*\mathcal{G}) \\
& \quad \downarrow \wr \qquad \text{proj. formula} \\
& Rf_*(Rp_*(p^*\mathcal{F} \otimes q^*\mathcal{G})) \\
& \quad \downarrow \wr \\
& Rh_*(p^*\mathcal{F} \otimes q^*\mathcal{G}). \qquad \boxed{}
\end{aligned}
$$

Rem: · Works for arbitrary S.

· g proper is enough

· or ℓ invertible on S + f smooth.

Cycle class maps

X non-singular variety $/k$ field of char $p \neq \ell$.

$C^r(X)$ = free abelian gp on prime cycles of codim r

Will define

$$cl^r : C^r(X) \to H^{2r}(X, \Lambda(r))$$

$$(\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell).$$

functorial, $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$

$$\begin{tikzcd}
cl^1 : C^1(X) \arrow[r] \arrow[d] & H^2(X, \Lambda(1)) & \mu_{\ell^n} \arrow[l] \\
\mathrm{Pic}(X) = H^1(X, \mathbb{G}_m) \arrow[ur, "\kappa"'] & &
\end{tikzcd}$$

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \to 1$$

<u>Defn</u> For Z nonsingular of codim r,

$$\begin{array}{rl} cl^r(Z) : \text{image } 1 \in & H^0(Z, \Lambda) \\ & \downarrow s \\ \text{purity} \nearrow & H^{2r}_Z(X, \Lambda(r)) \\ & \downarrow \\ & H^{2r}(X, \Lambda(r)) \end{array}$$

extend by linearity.

<u>Lemma</u> $Z \subseteq X$ codim r. Then

$H^s_Z(X, \Lambda) = 0$ for $s < 2r$.

$$H^{2r}_Z(X, \Lambda) = H^{2r}_{Z \setminus Z^{sing}}(X \setminus Z^{sing}, \Lambda)$$

$Z \setminus Z^{sing} \subseteq X \setminus Z^{sing}$ is a smooth pair.

<u>Pf</u> Induction on dim of Z.

Get cl by extending linearly.

Rem $cl : C^r(X_{\bar{k}}) \to H^{2r}(X_{\bar{k}}, \Lambda(r))$

- map is Galois equivariant.

$$\text{Cycles defined}/k \longmapsto H^{2r}(X_{\bar{k}}, \Lambda(r))^{G_k}$$

Conj (Tate) X sm. proj/k - f.g. field

$$C^r(X) \otimes \mathbb{Q}_\ell \to H^{2r}(X_{\bar{k}}, \mathbb{Q}_\ell(r))^{G_k}$$

is surjective.

Ex X, Y sm. proj. varieties of dim^n

$$\alpha \in H^{2n}(X \times Y_{\bar{k}}, \mathbb{Q}_\ell(n))^{G_k} \neq 0$$

$\Rightarrow \exists$ cycle on $X \times Y$ w/ non-trivial cycle class α (up $\mathbb{Q}_\ell$-linear combination)

Next time: Chern classes (another POV on cycle classes.