Étale Cohomology - 10/27/2020

Last time: étale π_1, $\mathbb{Z}_\ell$-sheaves, $\mathbb{Q}_\ell$-sheaves

Smooth Base Change

$$\begin{tikzcd} X' = X \times_S T \arrow[r, "f'"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

f.t. separated maps

(smooth base change theorem)

Thm $\mathcal{F}$ constructible sheaf on X, $(\mathcal{F})_{\bar{x}}$ order invertible on the base, f smooth.

Then $f^*(R^r\pi_*\mathcal{F}) \to R^r\pi'_* f'^*\mathcal{F}$

is an isomorphism

Rem Still applies when f is an inverse limit of sm. morphisms.

- f is the inclusion of a generic pt
- fraction field of strict henselization.

...

Thm (smooth + proper base change theorem)

Suppose π is sm. + proper, $\mathcal{F}$ lcc sheaf on X, $\#(\tilde{\mathcal{F}})_{\bar{x}}$ invtble on S

$$\begin{array}{c} X \\ \downarrow{\scriptstyle \pi} \\ S \end{array}$$

Then $R^r\pi_*\tilde{\mathcal{F}}$ are lcc.

<u>Rem</u> (exercise) Find a counterexample if you drop hypothesis on the orders of the fibers.

<u>Hint</u>: Family of ell. curves / char $k = p$, s.t. gen fiber is ordinary, but at least one fiber is supersingular, $\tilde{\mathcal{F}} = \mathbb{F}_p$.

<u>Claim</u> $(R^1\pi_*\mathbb{F}_p)_{\bar{x}} = \begin{cases} \mathbb{F}_p & \text{if } E_{\bar{x}} \text{ ordinary} \\ 0 & \text{if s.s.} \end{cases}$

proper base change

$$H^1(E_{\bar{x}}, \mathbb{F}_p) = \mathrm{Hom}_{cts}(\pi_1^{et}(E_{\bar{x}}), \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{ordinary} \\ 0 & \text{s.s.} \end{cases}$$

<u>Claim</u> $\pi_1^{et}(E_{\bar{x}}) = \prod_\ell T_\ell(E_{\bar{x}})$.

(exercise)

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X - scheme, $\bar\eta$, $\bar\xi$ - geometric pts of X,

$\bar\eta \rightsquigarrow \bar\xi$.

<u>Claim</u> $\exists$ non-canonical specialization map

$$\tilde{\mathcal{F}}_{\bar\xi} \to \mathcal{F}_{\bar\eta}$$ where $\mathcal{F}$ constructible sheaf on $X_{ét}$.

<u>Construction</u>

$$\begin{tikzcd} \bar\eta \arrow[r, squiggly] \arrow[dr, "\mathrm{im}(u)"] & \bar\xi \arrow[d] \\ U \arrow[u, dashed] \arrow[r] & X \end{tikzcd}$$

$$\tilde{\mathcal{F}}_{\bar\xi} = \varinjlim_{(U,\bar u)} \tilde{\mathcal{F}}(U)$$

For each $(U, \bar u)$, pick $\bar v \in U$ lifting $\bar\eta$. (compatibly)

$$\downarrow$$

$$\mathcal{F}_{\bar\eta} = \varinjlim_{(V,\bar v)} \mathcal{F}(V)$$

Depends on a choice of map $\mathcal{O}_{X,\bar\xi} \to \mathcal{O}_{X,\bar\eta}$.

above construction applied to $\mathcal{O}_X^{ét}$.

In practice, R dvr, k-residue field, K fraction field

$$\begin{array}{c} X \\ \downarrow \pi \text{ proper morphism} \\ \operatorname{Spec} R \end{array}$$

$$\tilde{\mathcal{F}} = R^i\pi_*(\underline{\mathbb{Z}/\ell}) \qquad \begin{array}{ccc} (\tilde{\mathcal{F}})_k = H^i(X_k, \mathbb{Z}/\ell) \\ \downarrow \qquad\qquad \downarrow \\ (\tilde{\mathcal{F}})_K = H^i(X_K, \underline{\mathbb{Z}/\ell}). \end{array}$$

Prop Constructible sheaf $\tilde{\mathcal{F}}$ satisfies

$\tilde{\mathcal{F}}$ lcc $\Longleftrightarrow$ all cospecialization maps are isomorphisms.

(Exercise).

Cor (Smooth + proper base change theorem)

$\pi \begin{array}{c} X \\ \downarrow \\ S \end{array}$ sm. proper, $\bar{\mathcal{F}}$ lcc sheaf on $X_{\acute{e}t}$ s.t. $\#(\bar{\mathcal{F}}_{\bar{x}})$ invertible on S,

Given $\bar{\eta}, \bar{\xi}$ geom. pts of S, $\bar{\eta} \rightsquigarrow \bar{\xi}$.

$$H^i(X_{\bar{\xi}}, \bar{\mathcal{F}}|_{X_{\bar{\xi}}}) \xrightarrow[\text{cosp}]{\sim} H^i(X_{\bar{\eta}}, \bar{\mathcal{F}}|_{X_{\bar{\eta}}})$$

is isom.

Cor k field of characteristic $p>0$, X/k sm. proper variety.
Then if X lifts to char 0, can compute its $\mathbb{F}_\ell / \mathbb{Z}_\ell$-cohomology, $\ell \neq \operatorname{char} k$.

(1) Lift to char 0:
$\exists$ sm. proper R-scheme $\mathcal{X}$ / dvr R
s.t. $R/\mathfrak{m}_R = k$, s.t. $\mathcal{X}_k \simeq X$.

Ex If $H^2(X, T_X) = 0$, get a formal lift.
Not enough!! If you can also formally lift ample line bundle $\xrightarrow{\text{formal GAGA}}$ lift.
In particular X projective, $H^2(X, T_X) = 0$, $H^2(X, \mathcal{O}_X) = 0$, you win.

Ex Curves, AVs, K3s (Deligne)
Hypersurfaces + complete intersections in $\mathbb{P}^n$.

(2) "Can compute cohomology."

Sm. + proper base change theorem: enough to compute $X_{\bar{K}}$. (K fraction field of a dvr to which X lifts)

$$\begin{tikzcd} \bar{K} & & \mathbb{C} \\ & \overline{\mathbb{Q}(\text{coeffs of polys defining } X)} \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

(1)
$$\begin{tikzcd} X_{\bar{K}} \arrow[dr] & & X_{\mathbb{C}} \arrow[dl] \\ & X_{\overline{\mathbb{Q}(\text{coeffs})}} & \end{tikzcd}$$

induce isos on coh.

(2) Cohom. of $X_{\mathbb{C}}$ is computable. (Artin comparison)

Prop $X/_k$ variety, k alg. closed, $k \subseteq L$ extn of alg. closed fields. Then if $\mathcal{F}$ is constr. sheaf on X_{et} s.t. $\#(\mathcal{F}_{\bar{x}})$ are invertible in k,

$$H^i(X_{et}, \mathcal{F}) \xrightarrow{\sim} H^i(X_{L,et}, \mathcal{F}|_{X_L}) \text{ is iso.}$$

Pf Smooth base change theorem.

Künneth formula + cycle class maps.

Thm X, Y proper k-schemes, k alg. closed

$\mathcal{F}$ const. sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on $X_{ét}$, $\mathcal{G}$ constructible sheaf on $Y_{ét}$.

$$R\Gamma(X_{ét}, \mathcal{F}) \overset{L}{\otimes} R\Gamma(Y_{ét}, \mathcal{G}) \to R\Gamma((X\times Y)_{ét}, \mathcal{F} \overset{L}{\otimes} \mathcal{G}) \quad \text{in } (D^b(Ab))$$

is an isom.

Cor

$$\bigoplus_{i+j=s} \mathrm{Tor}_{-r}^{\mathbb{Z}/\ell^n\mathbb{Z}}(H^i(X_{ét}, \mathcal{F}), H^j(Y_{ét}, \mathcal{G})) \Downarrow H^{r+s}((X\times Y)_{ét}, \mathcal{F}\otimes\mathcal{G})$$

if $\mathcal{F}$ or $\mathcal{G}$ is a sheaf of flat $\mathbb{Z}/\ell^n\mathbb{Z}$-modules.

Cor $/\mathbb{Z}_\ell$

$$0 \to \bigoplus_{r+s=m} H^r(X_{ét}, \mathbb{Z}_\ell) \otimes H^s(X_{ét}, \mathbb{Z}_\ell) \to H^m((X\times Y)_{ét}, \mathbb{Z}_\ell)$$

$$\bigoplus_{r+s=m+1} \operatorname{Tor}_1^{\mathbb{Z}_\ell}\left(H^r(X_{et}, \mathbb{Z}_\ell), H^s(X_{et}, \mathbb{Z}_\ell)\right).$$

$$0$$

<u>Non-ex</u> $X = Y = \mathbb{A}^1 / k$ - alg. closed field of char $p > 0$

Künneth is false in this case, for H^1

<u>Map</u>: $H^r(X_{et}, \mathcal{F}) \otimes H^s(Y_{et}, \mathcal{G}) \to H^{r+s}((X \times Y)_{et}, \mathcal{F} \otimes \mathcal{G})$

$[f] \in H^r(X_{et}, \mathcal{F})$, $[g] \in H^s(Y_{et}, \mathcal{G})$

on Čech cohomology.

$\{U\} \to X$ cover, $\{V\} \to Y$ cover, $\{U \times V\} \to X \times Y$ cover

$$(f, g) \longmapsto f \otimes g.$$