Étale Cohomology - 10/22/2020

Last time: $\pi_1^{\text{ét}}$

$$\{\text{lcc sheaves of Ab. gps on } X_{\text{ét}}\} \xleftarrow{\sim} \{\text{finite discrete } \pi_1^{\text{ét}}\text{-modules}\}$$

$$\widetilde{\mathcal{F}}_M \longleftarrow M$$

$$\mathcal{F} \longmapsto \begin{matrix}\text{rep'd by}\\ \text{a finite étale}\\ X\text{-scheme}\end{matrix} \longmapsto \text{finite } \pi_1^{\text{ét}}\text{-set}$$

$$\mathrm{Hom}(-, Y_M) \longleftarrow \begin{matrix}\text{fin. étale}\\ X\text{-scheme}\\ Y_M\end{matrix} \longleftarrow \pi_1^{\text{ét}}\text{-module}$$

<u>Prop</u> $H^i_{\text{cts}}(\pi_1^{\text{ét}}, M) \to H^i(X_{\text{ét}}, \widetilde{\mathcal{F}}_M)$ induces iso for $i = 0, 1$ (X conn'd).

<u>Pf</u> $X_{\text{ét}} \xrightarrow{\pi} \mathrm{FÉt}(X)$

<u>Claim</u> $\mathrm{FÉt}(X) \simeq$ Finite cts $\pi_1^{\text{ét}}$-sets

$\mathrm{Sh}_{\text{fin. ab. gps}}(\text{Finite cts } \pi_1^{\text{ét}}\text{-sets}) = \{\text{cts discrete finite } \pi_1^{\text{ét}}\text{-modules}\}$

$$H^i(\mathrm{F\acute{E}t}(X), M) \xrightarrow{\pi^*} H^i(X_{\acute{e}t}, \pi^* M)$$

$R^i \pi_* \pi^* M = 0$ for $i = 1$. $\boxed{}$

Cor $H^1(\pi_1^{\acute{e}t}(X, \bar{x}), M) = H^1(X_{\acute{e}t}, \mathcal{F}M)$

Finiteness Theorem

Thm X variety/k - separably closed field

$\hat{\mathcal{F}}$ constructible sheaf on $X_{\acute{e}t}$. Then

(1) if X proper, or

(2) if the stalks of $\mathcal{F}$ have order prime to $\mathrm{char}(k)$, then

$H^r(X_{\acute{e}t}, \hat{\mathcal{F}})$ is finite.

Pf (1) Proper base change

(2) Sketch proof.

Induction on dim'n.

(0) True in dim'ns 0, 1.

(1) Assume X smooth (simplifying assumption)

(2) Assume $U: X$ fits into an

elementary fibration:

fiberwise dense

$$\begin{tikzcd} U \arrow[r, hook, "\text{open}", "i"'] \arrow[dr, "f"'] & Y \arrow[d, "h", "\substack{\text{sm. proper}\\ \text{of rel. dim } 1}"'] & Z = Y \setminus U \arrow[l, hook', "\text{cl}"'] \arrow[dl, bend left, "\text{finite étale}"] \\ & S & \end{tikzcd}$$

(3) Devissage to reduce to the case $\tilde{\mathcal{F}}$ is lcc.

(4) $H^i(S, R^j f_* \tilde{\mathcal{F}}) \Rightarrow H^{i+j}(U, \tilde{\mathcal{F}})$

ETS by induction on dim'n that $R^j f_* \tilde{\mathcal{F}}$ is constructible for $j \geq 0$.

(5) $f = h \circ i$

$$R^s h_* R^t i_* \mathcal{F} \Rightarrow R^{s+t} f_* \mathcal{F}.$$

(1) h proper morphism

Proper base change $\Rightarrow$ ETS

$R^j h_* \mathcal{F}$ are constructible.

(6)
$$\begin{tikzcd} U \arrow[r, hook, "\text{open}", "i"'] \arrow[dr] & Y \arrow[d, "\substack{\text{sm. proper}\\ \text{curve}}"] & Z = Y/U \arrow[l, hook', "\text{cl}"', "j"] \arrow[dl, "\text{fin étale}"] \\ & S & \end{tikzcd}$$

(in case $\tilde{\mathcal{F}}$ actually constant)

$\tilde{\mathcal{F}} = \underline{\Lambda}$.

$$\iota_* \tilde{\mathcal{F}} = \iota_* \underline{\Lambda}$$
$$= \underline{\Lambda}_Y$$

(exercise) purity

$$R^1 \iota_* \underline{\Lambda} = j_* \underline{\Lambda} (?)$$

(purity) (uses order prime to char.)

$$R^i \iota_* \underline{\Lambda} = 0 \text{ for } i > 1$$

$\boxtimes$

Sheaves of $\mathbb{Z}_\ell$-modules

Defn $(\mathcal{M}_n, f_{n+1} : \mathcal{M}_{n+1} \to \mathcal{M}_n)$ is a sheaf of $\mathbb{Z}_\ell$-modules if

(a) each $\mathcal{M}_n$ is a constructible sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules

(b) f_{n+1} induces an iso

$$\mathcal{M}^{n+1}/\ell^n \mathcal{M}_{n+1} \xrightarrow{\sim} \mathcal{M}_n .$$

<u>Motivation</u> Given a ℓ-complete $\mathbb{Z}_\ell$-module N, $N = \varprojlim_n N/\ell^n N$

$$\wr$$

$$N \overset{\sim}{\longleftrightarrow} (N/\ell^n N, \text{ transition maps})$$

<u>Ex</u> $M_n = \underline{\mathbb{Z}/\ell^n\mathbb{Z}}$, $f_{n+1} : \underbrace{\mathbb{Z}/\ell^{n+1}\mathbb{Z} \to \mathbb{Z}/\ell^n\mathbb{Z}}_{\text{qt map.}}$

<u>Defn</u> (flat sheaf of $\mathbb{Z}_\ell$-modules)

in addition to the above:

$$0 \to M_s \xrightarrow{\ell^n} M_{n+s} \to M_n \to 0$$

exact.

<u>Motivation</u> This exactness characterizes flat ℓ-complete $\mathbb{Z}_\ell$-modules.

<u>Defn</u> (M, f_{n+1}) sheaf of $\mathbb{Z}_\ell$-modules

$$H^r(X_{et}, M) = \varprojlim H^r(X_{et}, M_n)$$

$$H^r_c(X_{ét}, M) = \varprojlim_n H^r_c(X_{ét}, M_n).$$

<u>Ex</u> X sm. proper curve of genus g / k - seplly closed and has char not equal to ℓ.

$$H^i(X_{ét}, \mathbb{Z}_\ell) = \varprojlim H^i(X_{ét}, \mathbb{Z}/\ell^n\mathbb{Z}) = \begin{cases} \mathbb{Z}_\ell & i=0 \\ T_\ell(\mathrm{Jac}\,X)(-1) & i=1 \leftarrow \mathbb{Z}_\ell^{2g} \\ \mathbb{Z}_\ell(-1) & i=2 \\ 0 & i>2 \end{cases}$$

<u>Thm</u> M flat sheaf of $\mathbb{Z}_\ell$-modules on a variety X/k - seplly closed. If X proper or $\ell \neq \mathrm{char}\, k \Rightarrow$

(a) $H^r(X_{ét}, M)$ f.g. $\mathbb{Z}_\ell$-module

(b) LES:

$$\cdots H^{r-1}(X_{ét}, M_n) \to H^r(X_{ét}, M) \xrightarrow{\ell^n} H^r(X_{ét}, M)$$

$H^n(X_{\bar{k}}, M_n) \to \cdots$

LES associated to "SES"

$$0 \to M \xrightarrow{\ell^n} M \to M_n \to 0$$

<u>Pf</u> (exercise)

(1) Reduce to the previous finiteness theorem

(2) Build LES above out of LES in cohomology arising from

$$0 \to M_s \xrightarrow{\ell^n} M_{n+s} \to M_n \to 0$$

by taking limit as $s \to \infty$. ⊠

<u>Defn</u> A $\mathbb{Z}_\ell$-sheaf (M, π_{n+1}) is <u>locally const.</u> if each M_n is locally constant.

Lisse: flat + locally constant.

$\hookrightarrow$ $(\mathcal{M}, f_{n+1})$ is locally constant $\not\Rightarrow$ $\exists$ cover of X s.t. it is constant.

π_1^{et}-reps associated to locally constant $\mathbb{Z}_\ell$-sheaves:

Suppose $\mathcal{M} = (\mathcal{M}_n, f_{n+1})_n$ is a locally const. $\mathbb{Z}_\ell$-sheaf.

$$\rho_n : \pi_1^{et}(X, \bar{x}) \xrightarrow{cts} \mathrm{Aut}(\mathcal{M}_n, \bar{x}).$$

$$\mathrm{Aut}(\mathcal{M}_{n+1}, \bar{x})$$

$\{$loc. const $\mathbb{Z}_\ell$-sheaves$\}$ $\leftrightarrow$ cts reps of π_1^{et} on f.g. $\mathbb{Z}_\ell$-modules.

$\{$lcc $\mathbb{Z}_\ell$-sheaves$\}$ $\leftrightarrow$ cts reps of π_1^{et} on f.g. flat (free) $\mathbb{Z}_\ell$-modules.

$\mathbb{Q}_\ell$-sheaves:

A $\mathbb{Q}_\ell$-sheaf: $\mathbb{Z}_\ell$-sheaf

A morphism of $\mathbb{Q}_\ell$-sheaves is

$$
\begin{tikzcd}
 & \mathcal{M}' \arrow[dl, "\text{finite kernel \& cokernel}"'] \arrow[dr] & \\
\mathcal{M} & & \mathcal{N}
\end{tikzcd}
$$

<u>Given:</u> a $\mathbb{Q}_\ell$-sheaf $\mathcal{M}$

$$H^i(X_{\acute{e}t}, \mathcal{M}) = \left(\varprojlim_n H^i(X_{\acute{e}t}, \mathcal{M}_n)\right) \otimes \mathbb{Q}_\ell$$

$$H^i_c(X_{\acute{e}t}, \mathcal{M}) = \left(\varprojlim_n H^i_c(X_{\acute{e}t}, \mathcal{M}_n)\right) \otimes \mathbb{Q}_\ell$$

Given a $\mathbb{Q}_\ell$-sheaf whose underlying $\mathbb{Z}_\ell$-sheaf is locally constant $\rightsquigarrow$ cts rep.

$$\rho : \pi_1^{\acute{e}t}(X, \bar{x}) \to GL_n(\mathbb{Q}_\ell)$$

This is an equivalence of categories.

Given ρ as above:

<u>Claim:</u> ρ conjugate to rep'n into $GL_n(\mathbb{Z}_\ell)$ (exercise)

Pf sketch $\mathbb{Z}_\ell^n \subseteq \mathbb{Q}_\ell^n$

topological fact: stabilizer of $\mathbb{Z}_\ell^n$ in $\pi_1^{\text{ét}}$ is open hence finite index.

$\sum_{g \in \pi_1/\text{Stab}} g\,\mathbb{Z}_\ell^n \leftarrow$ stable under $\pi_1^{\text{ét}}$

f.g. $\Rightarrow \cong \mathbb{Z}_\ell^n$.

$\square$