Étale Cohomology - 10/20/20

Last time: $\pi_1^{ét}$.

X - normal variety$/k$

$\mathrm{FÉt}(X) = \begin{cases} \text{Os: } Y \to X \text{ fin étale} \\ \text{morphisms are morphisms } /X \end{cases}$

Given a geometric pt $\bar{x} \to X$,

fiber functor $\to$ $F_{\bar{x}} : \mathrm{FÉt}(X) \to \mathrm{Set}$

$Y/X \longmapsto Y_{\bar{x}}$

$$\pi_1^{ét}(X, \bar{x}) = \mathrm{Aut}(F_{\bar{x}})$$

<u>Thm</u> (SGA 1)

$\mathrm{FÉt} \xrightarrow{F_{\bar{x}}}$ Finite cts $\pi_1^{ét}$-sets

is an equivalence of categories.

<u>Cor</u> $\mathbb{A}^1_k$, char k is positive, $\pi_1^{ét}(\mathbb{A}^1_k, \bar{x})$ is not topologically f.g.

<u>Pf</u> $H^1(\mathbb{A}^1_k, \mathbb{F}_p)$ not f.g.

Claim $\mathrm{Hom}_{cts}(\pi_1^{\acute{e}t}, \mathbb{F}_p) \xrightarrow{\sim} H^1(\mathbb{A}^1_k, \mathbb{F}_p)$
$\{\mathbb{F}_p\text{-torsors}\}$

Pf $\mathbb{F}_p \circlearrowright \mathbb{F}_p$ by addition

$\Rightarrow \mathrm{Hom}_{cts}(\pi_1^{\acute{e}t}(\mathbb{A}^1_k), \mathbb{F}_p) \xrightarrow{\sim}$ {finite cts $\pi_1^{\acute{e}t}$-sets s.t. the action factors through a map $\pi_1^{\acute{e}t}(\mathbb{A}^1_k) \to \mathbb{F}_p$.}

$\xrightarrow{\sim}$ $\mathbb{F}_p$-torsors. $\boxed{\Xi}$

Cor For any $\bar{x}_1, \bar{x}_2$ geom. pts of X,

$$\pi_1^{\acute{e}t}(X, \bar{x}_1) \simeq \pi_1^{\acute{e}t}(X, \bar{x}_2).$$

Pf (1) $\pi_1^{\acute{e}t}(X, \bar{x}_1)\text{-sets} \xleftarrow{\sim} \pi_1^{\acute{e}t}(X, \bar{x}_2)\text{-sets}$

$\searrow \mathrm{F\acute{E}t}(X) \swarrow$

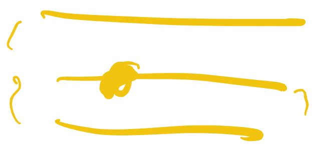
(2) (exercise) Category determines abstract gp. $\boxed{\Xi}$

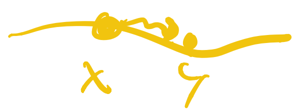
Rem In fact iso is well-defined up to inner conj.
To choose an iso, choose a sequence of specializations and generizations

$$\bar{x}_1 \rightsquigarrow \bar{y}_1 \rightsquigarrow \bar{y}_2 \leftsquigarrow \bar{y}_3 \rightsquigarrow \bar{y}_4 \cdots \rightsquigarrow \bar{x}_2$$

Claim If $\bar{x}$ specializes to $\bar{y}$, then

$$\begin{tikzcd} \mathrm{F\acute{e}t}(X) \arrow[d, bend right, "F_{\bar{x}}"'] \arrow[d, bend left, "F_{\bar{y}}"] \\ \mathrm{Set} \end{tikzcd} \qquad \mathcal{M}(Y/X)(z) = \bar{z} \cap Y_y.$$





Thm (Comparison Thm) X normal$/\mathbb{C}$, $\bar{x} \in X(\mathbb{C})$

$$\pi_1^{\acute{e}t}(X, \bar{x}) \longleftarrow \pi_1(X^{an}, \bar{x}^{an})$$

induces an iso

$$\pi_1(X^{an}, \bar{x}^{an})^{\wedge} \xrightarrow{\sim} \pi_1^{\acute{e}t}(X, \bar{x}).$$

Pf $\pi_1^{\acute{e}t}(X, \bar{x}) = \mathrm{Aut}(F_{\bar{x}})$

$\pi_1(X^{an}, \bar{x}^{an}) = \mathrm{Aut}(F_{\bar{x}}^{an} : \mathrm{Cov}(X) \to \mathrm{Set})$

ETS:

$$\begin{tikzcd} \mathrm{F\acute{E}t} \arrow[r, "an"] \arrow[d, "F_{\bar{x}}"] & \mathrm{Cov}(X) \arrow[dl, "F_{\bar{x}}^{an}"] \\ \mathrm{Set} & \end{tikzcd}$$

commutes

and an induces an equivalence

$$\mathrm{F\acute{E}t} \xrightarrow{\sim} \mathrm{FinCov}(X).$$

Riemann existence! $\square$

<u>Cor</u> X sm. proper curve of genus $g/\mathbb{C}$

$$\pi_1^{\acute{e}t}(X) = \langle a_1, b_1, \dots, a_g, b_g \mid \Pi [a_i, b_i] = 1 \rangle^{\wedge}$$

<u>Thm</u>

$$\begin{tikzcd} X_{\bar{k}} \arrow[r] & X \arrow[d] \\ & \mathrm{Spec}\, k \end{tikzcd}$$

$$1 \to \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\acute{e}t}(X, \bar{x}) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

short exact sequence

<u>Pf remarks</u> Surjectivity follows from geometric connectedness of X.

Specialization maps

X proper flat / complete dvr R w/ geom. conn'd fibers, $K = \mathrm{Frac}(R)$, $k = R/\mathfrak{m}$.

Thm Given $\bar{x} \to X_k$, the natural map

$$\pi_1^{ét}(X_k, \bar{x}) \to \pi_1^{ét}(X, \bar{x})$$

is an isomorphism.

Pf. Goal: $$\mathrm{FÉt}(X) \xrightarrow{\text{res}} \mathrm{FÉt}(X_k)$$

is an equivalence.

Essential surjective:

(1) Given $Y \to X_k$ fin. ét, construct $\mathcal{Y} \to \widehat{X}^{\mathfrak{m}}$ fin. étale.

Deformation theory:

$$\begin{tikzcd} Y_n \arrow[r, dashed] \arrow[d, "ét"'] & ? \arrow[d] \\ {} & {} \end{tikzcd}$$

$$\hat{X} \otimes R/\mathfrak{m}^n \hookrightarrow X \otimes R/\mathfrak{m}^{n+1}$$

$\mathcal{I}$ is the ideal defining this embedding.

$$\text{obs} \in \operatorname{Ext}^2_{X_K}(\underbrace{\Omega^1_{Y/X_K}}_{=0}, \mathcal{I}) = 0$$

If obs $= 0$, then $\exists\, Y_{n+1}$ flat over $X \otimes R/\mathfrak{m}^{n+1}$ making diagram Cartesian.

The set of such Y_{n+1} is a torsor for

$$\operatorname{Ext}^1_{X_K}(\underbrace{\Omega^1_{Y/X_K}}_{=0}, \mathcal{I}) = 0$$

Exercise (conormal exact sequence) $Y_{n+1}/X \otimes R/\mathfrak{m}^{n+1}$ is étale.

(2) $\exists!\; \mathcal{Y} \to \hat{X}^{\mathrm{an}}$ lifting Y.

Want: $\mathcal{Y} \to X$. (formal GAGA)

$\boxed{?}$

Cor Given X as above, $\bar{\eta} \to X_K$ geom. pt specializing to $\bar{\xi} \to X_k$, get

$$sp : \pi_1^{ét}(X_K, \bar\eta) \to \pi_1^{ét}(X_k, \bar\xi).$$

Pf $(X_K, \bar\eta) \to (X, \bar\eta)$

$$\begin{tikzcd}
\pi_1^{ét}(X_K, \bar\eta) \arrow[r] \arrow[drr, bend right] & \pi_1^{ét}(X, \bar\eta) \arrow[r, "\sim"] & \pi_1^{ét}(X, \bar\xi) \arrow[d, "s"] \\
& & \pi_1^{ét}(X_k, \bar\xi)
\end{tikzcd}$$

Thm X normal $\Rightarrow$ sp is surjective.

Pf Content: Given $Y \to X$ fin. étale w/ Y conn'd, then Y_K is also conn'd. $\square$

Cor X normal, flat proper$/R$, η, ξ alg. closure.

$$\pi_1^{ét}(X_{\bar K}, \bar\eta) \to \pi_1^{ét}(X_{\bar k}, \bar\xi)$$

is surjective.

Thm X variety$/k$ alg. closed of char 0,

L/k extn of algebraically closed fields,

$\pi_1^{ét}(X_L) \to \pi_1^{ét}(X)$ is an iso.

Pf Galois descent. (exercise)

Ex X sm. proper curve $/k = \bar{k}$ of char $p > 0$.
Then $\pi_1^{ét}(X, \bar{x})$ is topologically generated by at most $2\,\text{genus}(X)$ elts.

Pf (1) Lift to char 0 + algebraize.
(2) Surjective specialization map

$$\pi_1^{ét}(X_{\bar{K}}) \to \pi_1^{ét}(X)$$

can compute $/\mathbb{C}$

Thm (SGA 1) X as above
$$\pi_1^{ét}(X_{\bar{K}}) \to \pi_1^{ét}(X_{\bar{k}})$$
induces an isom. on prime-to-p completions, where $p = \text{char}(k)$.

Rem Analogous thms for non-proper varieties w/ snc compactification (Grothendieck-Murre) for "tame fundamental gp."

Prop $\{\text{lcc sheaves on } X_{\acute{e}t}\} \leftrightarrows \{\text{cts finite } \pi_1^{\acute{e}t}\text{-modules}\}$

$\widetilde{\mathcal{F}_M} \longleftarrow M$

Pf lcc sheaves are rep'd by finite étale covers.

(X conn'd)

Thm Canonical map

$$H^i_{cts}(\pi_1^{\acute{e}t}(X,\bar{x}), M)$$
$$\downarrow$$
$$H^i(X_{\acute{e}t}, \widetilde{\mathcal{F}_M})$$

induces an iso on H^0, H^1.

Pf $X_{\acute{e}t} \xrightarrow{t} F\acute{E}t(X)$

Claim $Sh(F\acute{E}t(X)) = \pi_1^{\acute{e}t}\text{-sets}$ $\Big\rangle$ exercise

$\mathcal{F}_M = f^*M.$

(

$R^1 f_* \mathcal{F}_M = 0.$ (torsors kill themselves)