Étale Cohomology - 10/15

Last time:

Thm X variety$/\mathbb{C}$

$$\begin{tikzcd} & X^{an\text{-}ét} \arrow[dl, "\pi"'] \arrow[dr, "an"] & \\ X^{an} & & X^{ét} \end{tikzcd}$$

(1) π^* induces an equivalence of categories
$$Sh(X^{an}) \to Sh(X^{an\text{-}ét})$$
(2) $an^* : H^i(X^{ét}, \mathcal{F}) \xrightarrow{\sim} H^i(X^{an\text{-}ét}, an^* \mathcal{F})$

constructible

Cor $\mathcal{F}$ constr. Abelian sheaf on $X^{ét}$,

Canonical iso

$$H^i(X_{ét}, \mathcal{F}) \xrightarrow{\sim} H^i(X^{an}, \mathcal{F}^{an})$$

$$\mathcal{F}^{an} = \pi_* an^* \mathcal{F}.$$

Pf of (1): Given $\tilde{\mathcal{F}} \in Sh(X^{an\text{-}\acute{e}t})$

want: $\tilde{\mathcal{F}}^{an} \in Sh(X^{an})$, isom.

$$\pi^* \tilde{\mathcal{F}}^{an} \xrightarrow{\sim} \widehat{\mathcal{F}}$$

$$(\pi^* G)^{an} \xrightarrow{\sim} G$$

$$\tilde{\mathcal{F}}^{an} := \pi_* \widetilde{\mathcal{F}}$$

Now: refine any an-ét cover to honest open cover.

Pf of (2) ($\tilde{\mathcal{F}}$ is lcc, X smooth)

Claim ETS:

(1) $\tilde{\mathcal{F}} \longrightarrow an_* an^* \widetilde{\mathcal{F}}$ isom.

(2) $R^i an_* an^* \widetilde{\mathcal{F}} = 0$ for $i > 0$.

Pf (Claim $\Rightarrow$ Thm)

$$H^{i+j}(X^{an\text{-}ét}, an^*\bar{\mathcal{F}}) \Leftarrow H^i(X_{ét}, R^j an_* an^* \bar{\mathcal{F}})$$

Claim $\Rightarrow H^i(X_{ét}, \bar{\mathcal{F}})$.

<u>Pf of (1)</u> ($\bar{\mathcal{F}} \to an_* an^* \bar{\mathcal{F}}$ is isom)

(if $\bar{\mathcal{F}}$ is lcc)

Local statement: WLOG $\bar{\mathcal{F}}$ is constant $\bar{\mathcal{F}} = \underline{\Lambda}$

$$\Gamma(U, \underline{\Lambda}) \to \Gamma(U^{an}, an^* \underline{\Lambda})$$
$$\xrightarrow{\sim} \Gamma(U^{an}, \underline{\Lambda})$$

ETS: $\pi_0(U^{an}) \to \pi_0(U)$ is a bijection

(i) Can assume U fits into an elementary fibration, conn'd

$$\begin{tikzcd} U \arrow[r, hook] \arrow[dr] & Y \arrow[d] & Z \arrow[l, hook'] \arrow[dl] \\ & S & \end{tikzcd}$$

(ii) Assume U^{an} (hence Y^{an})

is not conn'd. $Y = Y_1 \cup Y_2$

Y_i are unions of fibers of h. (prove the theorem for curves directly)

$h(Y_1), h(Y_2)$ are conn'd components of S.

Done by induction on dim'n.

Pf of (2) $R^i an_* an^* \tilde{\mathfrak{F}} = 0$ for $i > 0$.

(i) WLOG, $\bar{\mathfrak{F}}$ is constant, U fits into an elementary fibration:

Want: $R^i an_* \underline{\Lambda} = 0$ for $i > 0$

$an : U^{an\text{-}\acute{e}t} \to U_{\acute{e}t}$.

for U in elementary fibration.

(ii) <u>**Lemma**</u> U conn'd sm/$\mathbb{C}$, $\bar{\mathfrak{F}}$

lcc on $U^{an\text{-}ét}$, $r>0$.

Then for any $s \in H^r(U^{an\text{-}ét}, \mathcal{F})$
$\exists$ étale cover $\{U_i \to U\}$ s.t.
$s|_{U_i^{an}} = 0$.

<u>Cor</u> $(R^r an_* \mathcal{F})_{\bar{x}} = 0$
$\Rightarrow (R^r an_* \mathcal{F} = 0)$

(iii) Pf of lemma.

WLOG U sits in elementary fibration

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "f"'] & Y \arrow[d, "h"] & Z \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

$H^i(S, R^j f_* \mathcal{F}) \Rightarrow H^r(U^{an\text{-}ét}, \mathcal{F}) \ni s$

$R^j f_* \mathcal{F}$ = sheafification of $V \mapsto H^j(f^{-1}(V), \mathcal{F})$

By induction, can kill

contributions coming from

$R^j f_* \tilde{\mathcal{F}}$, $j > 0$.

Left w/ contributions from

$H^i(S, f_*^{an} \tilde{\mathcal{F}})$. lcc

Done by ind. hypothesis

(iii) Done except for base case, where $\dim U = 1$.

$\tilde{\mathcal{F}}$ lcc sheaf on $U^{an\text{-}et}$ ← Riemann surface

Want to kill $s \in H^i(U^{an\text{-}et}, \tilde{\mathcal{F}})$

for $i = 1, 2$ by passing to alg. étale covers of U.

$i=2$: Pass to any affine cover

$i=1$: $s \in H^1(U^{an\text{-}et}, \Lambda)$

Want: $\{U_i \to U\}$ étale cover s.t.

$s|_{U_i^{an\text{-}et}} = 0$.

s - corresponds to some Λ-torsor over $U^{an\text{-}ét}$ — covering space of U^{an} w/ Galois gp Λ.

<u>Claim</u> (Riemann existence thm)

$U_{fét} \xrightarrow{\sim} U^{f\text{-}an\text{-}ét}$ ← finite covering spaces.

<u>Rem</u> U projective — immediate from Serre's GAGA theorem.

$s \in H^1(U^{an\text{-}ét}, \Lambda) \rightsquigarrow U_s \to U^{an}$ fin. loc. anal. isom.

Riemann Existence $\Rightarrow$

$$\begin{array}{ccc} U_s & = & V^{an} \\ \downarrow & & \left(\begin{array}{c} V \\ \downarrow \\ U \end{array}\right) \\ U^{an} & & \end{array}$$

$s|_{V^{an}} = 0$ b/c torsors kill the corresponding coh. classes.

$\square$

Ex X K3 surface $/\mathbb{C}$

$$H^i(X_{et}, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & i=0,4 \\ 0 & i\neq 0,2,4 \\ (\mathbb{Z}/n\mathbb{Z})^{22} & i=2. \end{cases}$$

Non-ex $X = \mathbb{G}_m/\mathbb{C}$

$$H^1(X_{et}, \underline{\mathbb{Z}}) = 0$$

$$H^1(X^{an}, \mathbb{Z}) = \mathbb{Z}$$

Étale fundamental gps

X - locally Noetherian scheme.

$X_{fét} = $ Ob: $Y \to X$ fin. étale

Morphisms: morphisms$/X$.

Rem W/ covers topological covers, this is a site.

Defn $\bar{x}$ geom. pt of X

$$F_{\bar{x}} : X_{f\acute{e}t} \to \text{FinSet}$$ ← fiber functor

$$Y/X \longmapsto Y_{\bar{x}}.$$

$$\pi_1^{\acute{e}t}(X, \bar{x}) = \text{Aut}(F_{\bar{x}})$$

Topological gp w/ topology coarsest

s.t. $\pi_1^{\acute{e}t}(X,\bar{x}) \to \text{Aut}(F_{\bar{x}}(Y))$ ← discrete topology

is cts for all Y.

Ex $X = \text{Spec } k$.

$\bar{x} : \text{Spec } \bar{k} \to \text{Spec } k$

$X_{f\acute{e}t} : (\text{finite étale } k\text{-algebras})^{op}$

$$\pi_1^{\acute{e}t}(\text{Spec } k, \bar{x}) = \text{Gal}(k^s/k)$$

(exercise)

Ex $\mathbb{A}^1_k$, k char $p>0$ $\leftarrow$ not top. finitely generated

$$H^1(\mathbb{A}^1_{k,\acute{e}t}, \mathbb{F}_p) = \operatorname{coker}\left(k[t] \xrightarrow{x \mapsto x^p - x} k[t]\right)$$

$$\| $$

$$\operatorname{Hom}\left(\pi_1^{\acute{e}t}(\mathbb{A}^1_k, \bar{x})^{ab}, \mathbb{F}_p\right)$$

Ex E ell. curve $/k = \bar{k}$, char $k = p > 0$

$$\pi_1^{\acute{e}t}(E) = \varprojlim_n E[n] = \begin{cases} \mathbb{Z}_p \times \prod_{\ell \neq p} \mathbb{Z}_\ell^2 & E \text{ ord.} \\ \prod_{\ell \neq p} \mathbb{Z}_\ell^2 & E \text{ s.s.} \end{cases}$$

Ex X normal $/\mathbb{C}$, conn'd

$$\pi_1^{\acute{e}t}(X, \bar{x}) \xrightarrow{\sim} \pi_1(X^{an}, x)^{\wedge}$$

$\leftarrow$ profinite completion