Étale Cohomology - 10/13/2020

($\mathcal{F}$ order prime to char k)

Thm $Z \overset{cl}{\subseteq} X/_k$, X, Z smooth, Z of codim c in X. Then for $\tilde{\mathcal{F}}$ lcc

$$H^{r-2c}(Z, \tilde{\mathcal{F}}(-c)) \xrightarrow{\sim} H^r_Z(X, \mathcal{F})$$

is iso for $r \geq 0$.

Cor (Gysin sequence)

X, Z as above, $U = X \setminus Z$. Then

$0 \leq r < 2c-1$, the restriction map

($\mathcal{F}$ lcc) $\quad H^r(X_{\acute{e}t}, \mathcal{F}) \to H^r(U_{\acute{e}t}, \mathcal{F})$ is iso.

and LES

$$0 \to H^{2c-1}(X, \mathcal{F}) \to H^{2c-1}(U, \mathcal{F}|_U) \to H^0(Z, \mathcal{F}(-c))$$

$$\to H^{2c}(X, \mathcal{F}) \to H^{2c}(U, \mathcal{F}) \to H^1(Z, \mathcal{F}(-c)) \to \cdots,$$

Topological situation:

$$H^{2c-1+i}(U, \mathcal{F}|_U) \to H^i(Z, \mathcal{F}(-c))$$

$\tilde{Z}$: deleted neighborhood of Z

$\pi : \tilde{Z} \to Z$ homotopic to a sphere bundle

Leray s.s.:

$$\begin{array}{ccccccc} & & & & H^{2c-1+i}(U, \bar{F}) & & \\ & & & & \downarrow & & \\ \cdots \to H^{2c-1+i}(Z, \bar{F}) & \to & H^{2c-1+i}(\tilde{Z}, \bar{F}) & \to & H^{i}(Z, \bar{F}) & & \\ & & & & \downarrow & & \\ & & & & H^{2c+i}(Z, \bar{F}) \cdots & & \end{array}$$

(Thom-Gysin exact sequence)

__Cor__ (Gysin sequence)

X, Z as above, $U = X \setminus Z$. Then

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($\bar{F}$ lcc) $\quad H^r(X, \bar{F}) \to H^r(U, \bar{F})$ is iso.

and LES

$$0 \to H^{2c-1}(X, \bar{F}) \to H^{2c-1}(U, \bar{F}|_U) \to H^0(Z, \bar{F}(-c))$$

$$\to H^{2c}(X, \bar{F}) \to H^{2c}(U, \bar{F}) \to H^1(Z, \bar{F}(-c)) \to$$

<u>Pf</u> (Thm $\Rightarrow$ Cor)

In LES for coh. w/ supports, replace H_Z w/

$$H^{\bullet}(Z, \mathcal{F}(-c)) \qquad \square$$

<u>Ex</u> (Cohomology of projective space) ($k = \bar{k}$)

(1) $H^i(\mathbb{A}^1, \mu_n) = \begin{cases} \mu_n & i=0 \\ 0 & i>0 \end{cases}$ (check k X_n)

(2) (Kunneth) $H^i(\mathbb{A}^n, \mu_n) = \begin{cases} \mu_n & i=0 \\ 0 & i>0 \end{cases}$

(3) Gysin seq for $(\mathbb{P}^n, \mathbb{P}^{n-1})$ $c=1$

$$H^r(\mathbb{P}^n, \mathbb{Z}/n\mathbb{Z}) = H^r(\mathbb{A}^n, \mathbb{Z}/n\mathbb{Z})$$
$$0 \le r < 1$$

$$0 \to H^1(\mathbb{P}^n, \mu_n) \to H^1(\mathbb{A}^n, \mu_n) \to H^0(\mathbb{P}^{n-1}, \mathbb{Z}/n\mathbb{Z})$$

$$\to H^2(\mathbb{P}^n, \mu_n) \to H^2(\mathbb{A}^n, \mu_n) \to H^1(\mathbb{P}^{n-1}, \mathbb{Z}/n\mathbb{Z})$$

$$H^1(\mathbb{P}^n, \mu_n) = 0$$

$$H^i(\mathbb{P}^n, \mu_n) \simeq H^{i-2}(\mathbb{P}^{n-1}, \mathbb{Z}/n\mathbb{Z}) \text{ for } i \geq 2.$$

$$\overset{\text{induction}}{\Longrightarrow} \quad H^r(\mathbb{P}^n, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} (\mathbb{Z}/n\mathbb{Z})(-\frac{r}{2}) & r \text{ even}, \ r \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

Pf of purity (sketch)

$$U \underset{j}{\overset{\text{open}}{\hookrightarrow}} X \underset{i}{\overset{\text{cl}}{\hookleftarrow}} Z = X \setminus U$$

(i) Reduce to a local statement:

<u>Defn</u> $i^! \mathcal{F} = i^* \ker(\mathcal{F} \to j_* j^* \mathcal{F})$
$\in Sh(Z_{et})$

"sections of $\mathcal{F}$ supported on Z."

<u>Prop</u> i_* is left adjoint to $i^!$.

<u>Pf</u> exercise

<u>Cor</u> $i^!$ left exact, preserves injectives

<u>Pf</u> Exact left adjoint.

<u>Claim</u> (Local version of purity)

($Z, X, \tilde{\mathcal{F}}$ as in theorem)

$$R^{2c} i^! \tilde{\mathcal{F}} = i^* \tilde{\mathcal{F}}(-c)$$

$$R^r i^! \tilde{\mathcal{F}} = 0 \qquad r \neq 2c.$$

(2) Claim $\Rightarrow$ Thm

(i) $\Gamma(Z, i^! \tilde{\mathcal{F}}) = \Gamma_Z(X, \tilde{\mathcal{F}})$

(ii) (Grothendieck s.s.)

($R\Gamma \circ Ri^! = R\Gamma_Z$)

($i^!$ preserves injectives)

$$H^r(Z, R^s i^! \tilde{\mathcal{F}}) \Rightarrow H_Z^{r+s}(X, \tilde{\mathcal{F}})$$

By claim, $R^s i^! \tilde{\mathcal{F}} = 0$ for $s \neq 2c$

$$H^r(Z, i^* \tilde{\mathcal{F}}(-c)) = H_Z^{r+2c}(X, \tilde{\mathcal{F}}) \quad \square$$

(3) <u>Pf of claim</u>

(i) Reduce to $(\mathbb{A}^m, \mathbb{A}^{m-c})$

(ii) Induction on m, c

Base case: $m=1$, $c=1$, which we did last time.

Comparison Theorems (Artin comparison) + Elementary fibrations.

Defn (Elementary fibration)

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr, "f"'] & Y \arrow[d, "h"] & Z \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

(1) j Zariski-open, $j(U)$ is fiberwise dense in Y
$Z = Y \setminus U$, $i: Z \hookrightarrow Y$ cl embedding

(2) h sm. proj. w/ geom. irred. fibers, rel dim 1.

(3) g finite étale

Key: "Topology of the fiber of f are constant"

Prop (Artin) X sm$/k = \bar{k}$. For each $x \in X$
$\exists$ Zariski open $U \ni x$ s.t. U
fits into an elementary fibration.

Pf Pick a U, embed in $\mathbb{P}^n$, project down until the base has dim'n $\dim U - 1$. $\square$

Thm X variety$/\mathbb{C}$, $\mathcal{F} \in Sh^{\text{const, ab}}(X_{ét})$

$$\begin{tikzcd} & X^{an\text{-}ét} \arrow[dl, "\pi"'] \arrow[dr, "an"] & \\ X^{an} & & X_{ét} \end{tikzcd}$$

(exercise)
(1) π induces iso on coh. of all Abelian sheaves (induces eq. of categories)

$$\pi^* : Sh(X^{an}) \xrightarrow{\sim} Sh(X^{an\text{-}ét})$$

(next time)
(2) $an^* : H^i(X_{ét}, \mathcal{F}) \xrightarrow{\sim} H^i(X^{an\text{-}ét}, an^* \mathcal{F})$ is an isom.

X^{an} — site associated to Euclidean topology on X^{an}.

$X^{an\text{-}ét}$ — category of cpx-analytic spaces mapping to X^{an} via local analytic isos.

covers are covers

<u>Cor</u> For $\mathcal{F}$ as in the thm, there is a canonical iso

$$H^i(X_{et}, \mathcal{F}) \xrightarrow{\sim} H^i(X^{an}, \mathcal{F}^{an})$$

Will prove this for X smooth, $\mathcal{F}$ lcc. via elementary fibrations.