Étale Cohomology - 11/3/2020

Brauer-Manin obstructions:

Goal - $X/\mathbb{Q}$ variety; understand obstructions to rat'l pts of a non-local nature

Idea: $X(\mathbb{Q}) \longrightarrow X(\mathbb{A}_\mathbb{Q})^\alpha \longrightarrow X(\mathbb{A}_\mathbb{Q})$

$\alpha \in Br(X) = H^2(X_{ét}, \mathbb{G}_m)_{tors}$

$x: \operatorname{Spec} \mathbb{Q} \to X$

$$x^*\alpha \in Br(\mathbb{Q})$$

$$0 \to Br(\mathbb{Q}) \to \bigoplus_{v \text{ place of } \mathbb{Q}} Br(\mathbb{Q}_v) \xrightarrow{\Sigma \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0$$

$$Br(\mathbb{Q}_v) = \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{if } v \text{ finite} \\ \mathbb{Z}/2 & \text{if } v \text{ is real} \\ 0 & \text{if } v \text{ complex} \end{cases}$$

$$X(\mathbb{A}_\mathbb{Q})^\alpha = \{(x_v) \in \prod_v X(\mathbb{Q}_v) \mid \Sigma \mathrm{inv}_v(x_v^*\alpha) = 0\}$$

Non-obvious fact: $X(\mathbb{A}_{\mathbb{Q}})^{\alpha}$ is effectively computable.

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Last time: Künneth + cycle class maps

Today: Chern classes of vector bundles + Poincaré duality.

Chern classes: X sm. proj / field. (ℓ prime to characteristic)

Given a vector bundle $\mathcal{E}$ on X

construct $c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Z}_\ell(i))$

Ex $i=1$: $c_1(\mathcal{E}) \in H^2(X, \mathbb{Z}_\ell(1))$

$$\mathbb{Z}_\ell(1) : (\cdots \to \mu_{\ell^n} \to \mu_{\ell^{n-1}} \to \cdots)$$

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{z \mapsto z^{\ell^n}} \mathbb{G}_m \to 1$$

$$H^1(X, \mathbb{G}_m) = \operatorname{Pic} X \xrightarrow{\kappa_{\ell^n}} H^2(X, \mu_{\ell^n})$$

$$\operatorname{Pic} X \xrightarrow{\kappa} H^2(X, \mathbb{Z}_\ell(1))$$

<u>Defn</u> $c_1(\mathcal{E}) = \kappa(\det(\mathcal{E}))$

<u>Thm</u> $\exists!$ assignment $\mathcal{E} \mapsto c_i(\mathcal{E})$ s.t.

(1) Functorial under pullback

$$f^*(c_i(\mathcal{E})) = c_i(f^*(\mathcal{E}))$$

(2) $c_1(\mathcal{E}) = \kappa(\det(\mathcal{E}))$

(3) Multiplicative under exact sequence.

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots$$

$$\in \bigoplus_i H^{2i}(X, \mathbb{Z}_\ell(i))$$

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

$$\boxed{c(\mathcal{E}) = c(\mathcal{E}_1) \cdot c(\mathcal{E}_2).}$$

<u>Pf idea</u> Analyze cohomology of $\mathbb{P}(\mathcal{E})$ from which Chern classes can be extracted.

Uniqueness: Induction on rk,

$\mathbb{P}(\mathcal{E})$

$$\begin{array}{l} \mathbb{P}(\mathcal{E}) \\ \downarrow \pi \qquad 0 \to V \to \pi^*\mathcal{E} \to \mathcal{O}(1) \to 0 \\ X \end{array}$$

$$c(\pi^*(\mathcal{E})) = (1 + c_1(\mathcal{O}(1))) \cdot c(V)$$

<u>Claim</u> $\pi^*: H^*(X) \to H^*(\mathbb{P}(\mathcal{E}))$ injective.

$\boxed{\mathcal{E}}$

<u>Rem</u> Multiplicativity in short exact sequences

$$K^*(X) \xrightarrow{c} \bigoplus_i H^{2i}(X, \mathbb{Z}_\ell(i))$$

multiplicative map, where

$K^*(X) =$ free ab. gp on isom. classes of v.b. on $X / [\mathcal{E}] = [\mathcal{E}_1] + [\mathcal{E}_2]$ if $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ is a short exact sequence.

Can modify c to get a ring homomorphism

$$\chi: K^*(X) \to \bigoplus_i H^{2i}(X, \mathbb{Q}_\ell(i))$$

ring w/ $\otimes$ as multiplication.

ring w/ cup product

$$C^*(X)_{\mathbb{Q}} \xrightarrow[\sim]{ch^{-1}} K^*(X)_{\mathbb{Q}} \xrightarrow{\sim} \oplus H^{2i}(X, \mathbb{Q}_\ell(i))$$

Chow grps of X

Composition is precisely the cycle class map from before.

X sm. proj.: ch induces an isomorphism

$$C^*(X)_{\mathbb{Q}} \xrightarrow{\sim} K^*(X)_{\mathbb{Q}}$$

<u>Poincaré duality</u>:

X sm variety $/k = k^s$, $\Lambda = \underline{\mathbb{Z}/m\mathbb{Z}}$, m invtble in k

<u>Thm</u> $H^i(X_{\acute{e}t}, \Lambda) \times H_c^{2\dim X - i}(X_{\acute{e}t}, \Lambda(d))$ $\quad \Lambda \otimes \mu_m^{\otimes d}$

$$\downarrow \cup$$

$$H_c^{2\dim(X)}(X_{\acute{e}t}, \Lambda(d))$$

$$\wr\downarrow \mathrm{Tr}$$
$$\mathbb{Z}/m\mathbb{Z}$$

$\exists!$ Tr canonical isom, under Tr, $\cup$ is a perfect pairing.

Analogy in topology: X oriented mfld.

$$H_c^{\dim X}(X, \mathbb{R}) \xrightarrow{\int} \mathbb{R}$$

"Trace map"

$$\begin{array}{ccccc} H^i(X,\mathbb{R}) & \times & H_c^{\dim X - i}(X,\mathbb{R}) & \to & H_c^{\dim X}(X,\mathbb{R}) \\ \omega & & \eta & \longmapsto & \omega \wedge \eta \end{array}$$

Pf (1) Construction of Tr : $H^{2\dim X}(X, \Lambda(d)) \to \mathbb{Z}/m\mathbb{Z}$

(2) Checking that we have a perfect pairing.

Construction of Tr map:

Rem Tony Feng's notes are a good

reference.

More general claim: Suppose $f: X \to S$

sm. compactifiable morphism w/ geom. conn'd fibers.

<u>Claim</u> $\exists$ canonical $\mathrm{Tr}: R^{2d} f_! f^* \bar{\mathcal{F}}(d) \to \bar{\mathcal{F}}$

where d is rel dim'n of f.

<u>Recall:</u>
$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & X' \arrow[d, "\tilde{f}"] \\ & S \end{tikzcd}$$
cpctification of f

$$R^i f_! \mathcal{G} := R^i \hat{f}_* \iota_! \mathcal{G}.$$

s.t. (1) For curves / sep'bly closed field, agrees w/ computation we did earlier.

(2) Functorial in $\bar{\mathcal{F}}$

(3) Compatible w/ base change

(4) If f étale

$$f_! f^* \bar{\mathcal{F}} \to \bar{\mathcal{F}}$$

is counit of adjunction b/w $f_!$, f^*.

(5) Compatible w/ compositions.

<u>Pf Sketch:</u>

(1)
$$\begin{tikzcd} X \arrow[r, "\text{ét}", "g"'] \arrow[dr] & \mathbb{A}^n_S \arrow[d, "\pi"] \arrow[r] & \mathbb{A}^{n-1}_S \arrow[d] \\ & S & \mathbb{A}^{n-2}_S \arrow[l, "\cdots"'] \end{tikzcd}$$

Compatibility w/ compositions: enough to define Tr map for g, π.

g: Counit of adjunction $g_!, g^*$.

π: Factor into a sequence of relative curves, use the curve case.

$$\begin{tikzcd} & X' \arrow[d, "\pi", "\text{proper}"'] \\ X \arrow[ur, hook, "\text{open}"] \arrow[r, "g"'] & Y \end{tikzcd}$$

$g : X \to Y$ étale morphism

$$g_! g^* \mathcal{F} \to \mathcal{F}$$

(1) $U \subseteq Y$, want $g_! g^* \mathcal{F}(U) \to \mathcal{F}(U)$

$g_! = \pi_* \iota_!$

(i) $\pi_* \vdash \pi^*$ adjoint

(ii) $\iota_! \longmapsto \iota^*$

<u>Pf of (ii)</u> $\iota_! \iota^* \tilde{\mathcal{F}}(U) \xrightarrow{Tr} \tilde{\mathcal{F}}(U)$

(1) $U \subseteq \operatorname{im} \iota$

$Tr = id$

(2) $U \not\subseteq \operatorname{im} \iota$, $\iota_! \iota^* \tilde{\mathcal{F}}(U) = 0$

$Tr = 0$.

(i) $\pi_* \pi^* \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$

After pullback through π,

Want: $\pi^* \pi_* \pi^* \tilde{\mathcal{F}} \to \pi^* \tilde{\mathcal{F}}$

$\exists$ from adjointness of $\pi^* \vdash \pi_*$

Claim: map descends to a map

$\pi_* \pi^* \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$.

Counit of adjunction:

$F : C \rightleftarrows D : G$

F left adjoint to G.

$\eta : FG(X) \to X$

$\mathrm{id}_{G(X)} \in \mathrm{Hom}_C(G(X), G(X)) \simeq \mathrm{Hom}_D(FG(X), X)$

$\mapsto \eta$ "counit of adjunction"

$X \to GF(X)$ "unit of adjunction".

(2) Reduce construction to this case: by Mayer-Vietoris.

Well-definedness / Uniqueness

(1) Choice of open set U

(2) Choice of étale map

construction for "standard étale maps" forced by axioms.

$U \to \mathbb{A}^n_S$.

(1) Take common refinements

(2)
$$\begin{tikzcd} X \arrow[r, "\text{ét}"] \arrow[d, "\text{ét}"'] & \mathbb{A}^1_S \arrow[d] \\ \mathbb{A}^1_S \arrow[r] & S \end{tikzcd}$$
Case of curves.

<u>Claim</u> If $X \xrightarrow{f} S$ has geom. conn'd fibers

$R^{2d} f_! f^* \mathcal{F}(d) \to \mathcal{F}$ is an.

<u>Pf Sketch</u> (1) Case of relative curve, check on stalks

(2) Reduce to this case via structure theorem for étale morphisms.