Étale Cohomology - 11/24/2020

Last time: Grothendieck-Lefschetz trace formula

Upshot:

<u>Thm</u> U_0 sm curve $/\mathbb{F}_q$, $\mathcal{E}_0$ loc. const $\mathbb{Q}_\ell$-sheaf on U_0 $(\ell \nmid q)$, then

$$\sum_{x \in U^F} \mathrm{Tr}(F_x \mid \mathcal{E}_x) = \sum (-1)^r \mathrm{Tr}(F \mid H^r_c(U, \mathcal{E}))$$

$$S(U_0, \mathcal{E}_0, t) = \exp\left(\sum_m \sum_{x \in U^{F^m}} \frac{\mathrm{Tr}(F_x \mid \mathcal{E}_x) t^m}{m} \right)$$

Thm $\Rightarrow$ S is rat'l, can be written in terms of char. poly of F on $H^i_c(U, \mathcal{E})$.

<u>Today</u> Study $S(U_0, \mathcal{E}_0, t)$ for very special $\mathcal{E}_0$ arising from Lefschetz fibrations.

MAIN LEMMA] X_0 sm. affine genus 0 curve (geom. conn'd)/ $\mathbb{F}_q$, $X = (X_0)_{\overline{\mathbb{F}}_q}$

$\mathcal{E}$ - loc. const. $\mathbb{Q}_\ell$-sheaf on X_0, E - corresponding rep'n of $\pi_1^{et}(X_0)$. Assume:

(1) For each $x \in |X|$, $F_x \curvearrowright \mathcal{E}_x$ w/ char. poly $\in \mathbb{Q}[t]$.

(2) non-degenerate skew-symmetric form
$$\psi : E \times E \to \mathbb{Q}_\ell(-n)$$

(3) $\pi_1^{et}(X) \to GL(E)$ "big monodromy"
$$\rho \searrow \quad \cup\!\!\!\mid$$
$$Sp(E, \psi)$$

$\mathrm{im}(\rho)$ open in $Sp(E, \psi)$

Then:

(a) E has "wt" n

(eigenvalues α of $F_x \curvearrowright \mathcal{E}_x$ have abs value $q^{n(\deg x)/2}$ for any $x \in |X|$)

for any embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$

$\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$.

(b) $F \curvearrowright H^1_c(X, \mathcal{E})$ has rat'l char poly

and for all eigenvalues α,

$|\alpha| < q^{n/2+1}$ (for all embeddings $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$).

(c) $F \curvearrowright H^1(\mathbb{P}^1, j_* \mathcal{E})$ has rat'l

char poly, all eigenvalues α satisfy

$q^{n/2} < |\alpha| < q^{n/2+1}$.

Where does $\mathcal{E}$ come from?

<u>Defn</u> (Lefschetz pencil)

X sm. proj. variety, $X \xhookrightarrow{|\mathcal{L}|} \mathbb{P}^n$

$\ell \underset{\text{line}}{\subseteq} \check{\mathbb{P}}^n$ (linear family of hyperplanes H_t)

is a Lefschetz pencil if

(1) The base locus (or axis) of the pencil

$A = \bigcap_t H_t$ intersects X transversely.

(2) $X_t = X \cap H_t$ this is smooth for

t in a dense open U of ℓ

(3) For $t \notin U$, $X_t = X \cap H_t$ has a

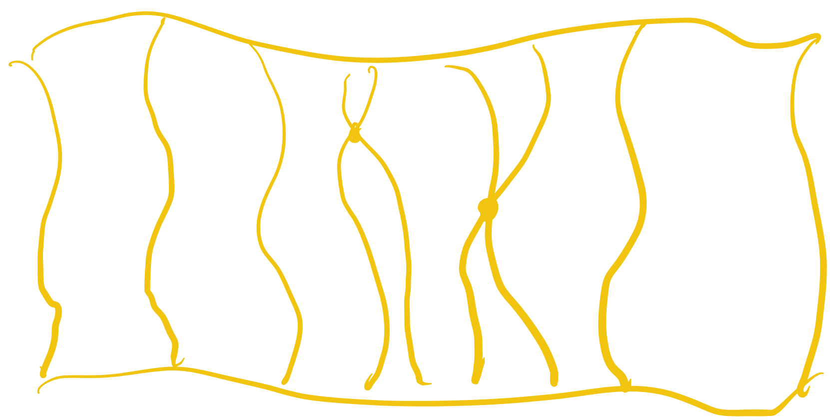
unique singular pt which is an ordinary double pt: $\widehat{\mathcal{O}}_{X,p} \cong k[[t_1, \dots, t_n]]/$ non-degenerate quadric.

Picture of

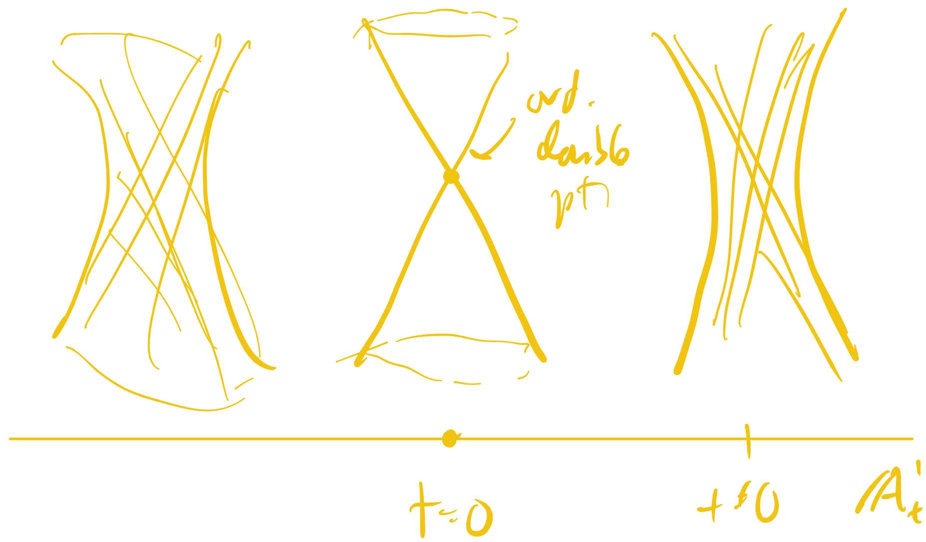
$$\begin{array}{c} \{X \times \ell \cap \mathcal{H}_p\} \\ \downarrow \\ \ell \end{array}$$

family of hyperplanes$/\ell$

fiber over t is X_t.



Ex $\{x^2+y^2+tz^2=0\} \subseteq \mathbb{P}^2_e / \mathbb{C}$



Thm (existence of Lefschetz pencils)

X sm. proj $/ k=\bar{k}$, $\mathcal{L}$ very ample line bundle on X. Then $\ell \subseteq |\mathcal{L}^{\otimes 2}|$ such that ℓ is a Lefschetz pencil.

Pf Bertini argument.

Strategy of proof of RH:

X_0 sm. proj. even-dim'l variety $/ \mathbb{F}_q$

$\dim X_0 = n+1$

want: eigenvalues of F on $H^{n-1}(X, \mathbb{Q}_\ell)$
satisfy $q^{n/2} \le |\alpha| \le q^{n/2+1}$

(1) WLOG can assume X admits a Lefschetz pencil which descends to X_0.

(2) Can replace X_0 w/

$$\underset{\substack{\text{base} \\ \text{locus of Lefschetz} \\ \text{pencil}}}{Bl} X_0 = (X \times \ell) \times_\ell \mathcal{H}$$
$$\downarrow$$
$$\ell$$

which is a family over $\mathbb{P}^1_t$ w/ fiber $(X_0) \cap H_t$.

Coh. of blowup built out of coh. of X, coh of $X \cap$ base locus.
codim 2

Key claim $Bl_{\substack{\text{base}\\ \text{locus}}} X \to X$ induces an injection $H^*(X) \to H^*(Bl)$.

Can assume $\begin{matrix} X \\ \pi \downarrow \\ \mathbb{P}^1 \end{matrix}$ "Lefschetz fibration"

(3) Enough to understand eigenvalues of Frobenius on

(a) $H^2(\mathbb{P}^1, R^{n-1}\pi_* \mathbb{Q}_\ell)$

(b) $H^1(\mathbb{P}^1, R^n \pi_* \mathbb{Q}_\ell)$

(c) $H^0(\mathbb{P}^1, R^{n+1}\pi_* \mathbb{Q}_\ell)$

(a) $H^2(\mathbb{P}^1, R^{n-1}\pi_* \mathbb{Q}_\ell)$ Fact: constant sheaf

$= H^{n-1}(X_t, \mathbb{Q}_\ell)(-1)$

X_t gen. fiber, $\dim X_t = n$

Let $Y \subseteq X_t$ be a sm. hyperplane section (exists by Bertini) $\dim Y = n-1$

Lefschetz hyperplane thm:

$$H^{n-1}(X_t, \mathbb{Q}_\ell) \hookrightarrow H^{n-1}(Y, \mathbb{Q}_\ell)$$

middle coh. of an even dim'l variety, win by induction hypothesis.

(b) $H^1(\mathbb{P}^1, R^n \pi_* \mathbb{Q}_\ell)$

$\hookleftarrow$ MAIN LEMMA (next time)

(c) $H^0(\mathbb{P}^1, R^{n+1} \pi_* \mathbb{Q}_\ell)$

$\hookleftarrow$ in good situations, this is a constant sheaf — win via application of Lefschetz + Poincaré duality. $\boxed{\lightning}$

Pf of <u>MAIN LEMMA.</u>

$\mathcal{E}_0$ lc $\mathbb{Q}_\ell$-sheaf on X_0 w/ $F_x \supseteq \mathcal{E}_x$ rat'l

skew-symmetric $\psi: E \times E \to \mathbb{Q}_\ell(-n)$, "big monodromy"

(a) E has wt n (eigenvalues of F_x, $x \in |X|$ have abs. value $q^{n(\deg x)/2}$).

<u>Lemma 1</u> $\left(\bigotimes_{2k} E\right)_{\pi_1(X)} = \mathbb{Q}_\ell(-kn)^{\oplus N}$

<u>Lemma 2</u> If $\forall k$,

$\zeta(X_0, \mathcal{E}_0^{\otimes 2k}, t)$ converges for $t < \frac{1}{q^{kn+1}}$, then E has wt n.

Lemmas $\Rightarrow$ MAIN LEMMA (a).

- Lemma 1 $\Rightarrow$ hyp of Lemma 2

$$\zeta(X_0, \mathcal{E}_0^{\otimes 2k}, t) = \frac{\text{poly} \quad H_c^1(X, \mathcal{E}^{\otimes 2k})}{\cancel{\det(1 - F^* t / H_c^0)} \det(1 - F^* t / H_c^2)}$$

$H_c^0(X, \mathcal{E}^{\otimes 2k}) = 0$ b/c X_0 affine

$$H_c^2(X, \mathcal{E}^{\otimes 2k}) \overset{PD}{=} H^0(X, (\mathcal{E}^\vee)^{\otimes 2k}(1))^\vee$$

$$= \left(((E^\vee)^{\otimes 2k})^{\pi_1}(1)\right)^\vee$$

$$= (E)^{\otimes 2k}_{\pi_1}(-1) \overset{\text{Lemma 1}}{=} \mathbb{Q}_\ell(-kn-1)^{\oplus N}$$

$$\zeta(X_0, \mathcal{E}_0, t) = \frac{\text{poly}}{(1 - q^{kn+1} t)^N}$$

satisfies hyp. of Lemma 2.