Étale Cohomology - 11/19/2020

Last time: • Discussed Frobenii
• Preliminary reductions
• Lefschetz trace formula

Lefschetz: - Non-proper varieties
- Non-constant sheaves
- Sheaves over $\mathbb{Z}/\ell^n\mathbb{Z}$

Non-constant sheaves $\mathcal{E}$ - sheaf on $X_{ét}$

$$\begin{tikzcd} \varphi^* : H^i(X_{ét}, \mathcal{E}) \arrow[r] \arrow[dr, bend left, "(\varphi, \varphi_{\mathcal{E}})^*"'] & H^i(X_{ét}, \varphi^*\mathcal{E}) \arrow[d, "\varphi_{\mathcal{E}}"] \\ & H^i(X_{ét}, \mathcal{E}) \end{tikzcd}$$

$\leftarrow$ need

$$\varphi_{\mathcal{E}} : \varphi^*\mathcal{E} \to \mathcal{E}.$$

Given $\varphi_{\mathcal{E}}$, get $(\varphi^*(\mathcal{E}))_x \xrightarrow{(\varphi_{\mathcal{E}})_x} \mathcal{E}_x$

$$\begin{tikzcd} (\varphi^*(\mathcal{E}))_x \arrow[r, "(\varphi_{\mathcal{E}})_x"] \arrow[d, equal] & \mathcal{E}_x \\ \mathcal{E}_{\varphi(x)} \arrow[ur, dashed, "\varphi_{\mathcal{E},x}"'] & \end{tikzcd}$$

If $x = \varphi(x)$, get $\varphi_{\mathcal{E},x} : \mathcal{E}_x \to \mathcal{E}_x$

Want: X_0 variety $/\mathbb{F}_q$, $X = (X_0)_{\overline{\mathbb{F}}_q}$

$$\sum_{x \in X^{\varphi}} \mathrm{Tr}(\varphi_x \mid \mathcal{E}_x) = \sum_i (-1)^i \left((\varphi, \varphi_{\mathcal{E}})^* \mid H^i_c(X_{\bar{s}}, \mathcal{E}) \right)$$

Ex X finite set

Q When is there a natural map $F_{\mathcal{E}} : F^* \mathcal{E} \to \mathcal{E}$?

A If $\mathcal{E}$ comes from a sheaf on X_0.

Pf Last time, proved $F_{abs} : X_{\acute{e}t} \to X_{\acute{e}t}$ is naturally isomorphic to the identity.

Cor Given $\mathcal{E}_0$ on $X_0/\mathbb{F}_q$,

$$F_{abs}^* \mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}_0$$

Restricting to $X_{\acute{e}t}$ gives desired $F_{\mathcal{E}}$. ∎

Slightly more explicit:

$$F_{abs}^{-1} U \xrightarrow{\sim} U$$

$$\downarrow \mathcal{E}(-)$$

$$\mathcal{E}(F_{abs}^{-1}(U)) \to \mathcal{E}(U)$$

$$\Downarrow$$

$$F_{abs *} \mathcal{E} \xrightarrow{\sim} \mathcal{E}$$

$$\underset{\text{adjunction}}{\overset{\wr}{}}\quad \mathcal{E} \to F^* \mathcal{E}.$$

<u>Target thm:</u> U_0 sm. curve$/\mathbb{F}_q$, $\mathcal{E}_0$ locally constant $\mathbb{Q}_\ell$-sheaf on U_0. Then

$$\sum_{x \in U^F} \mathrm{Tr}(F_x | \mathcal{E}_x) = \sum (-1)^r \,\mathrm{Tr}(F | H^r_c(U, \mathcal{E}))$$

<u>Defn</u> (ζ-fn of a sheaf)

$U_0, \mathcal{E}_0$ as above

$$\zeta(U_0, \mathcal{E}_0, t) = \exp\left(\sum_m \sum_{x \in U^{F^m}} \mathrm{Tr}(F_x^m | \mathcal{E}_x) \frac{t^m}{m} \right)$$

Target thm $\overset{+}{}$ finiteness thms for étale coh. $\Rightarrow \zeta(U_0, \mathcal{E}_0, t)$ is always rat'l.

<u>Ex</u> · $\mathcal{E}_0 = \mathbb{Q}_\ell$, $\zeta(U_0, \mathbb{Q}_\ell, t)$ is $\zeta_{U_0}(t)$

· $X_0 \xrightarrow{\pi} U_0$ sm. proper morphism $\quad \mathcal{E}_0^i = R^i \pi_* \mathbb{Q}_\ell$

$$\prod_i \zeta(U_0, \mathcal{E}_0^i, t)^{(-1)^i} = \zeta_{X_0}(t)$$

by Leray spectral sequence for π.

<u>Rem</u> (1) Wrote target thm in terms of H^i_c, but using PD could have written it in terms of $H^{2-i}(X, \mathcal{E}^\vee(1))$.

(2) Formula can re-interpreted in terms of π_1 (true for curves)

$$\{\text{lisse } \mathbb{Q}_\ell\text{-sheaves on } U_0,\ \mathcal{E}\}$$
$$\text{cts} \updownarrow$$
$$\{\pi_1^{et}(U_0, u)\text{-repns into } GL(\mathcal{E}_u)\}$$

LHS: $\sum_{x \in U^F = U_0(\mathbb{F}_q)} Tr(F_x \mid \mathcal{E}_x)$ ← can interpret this as $Tr(F_x \in \pi_1^{et}(U_0))$

Given $x \in U_0(\mathbb{F}_q)$ get $\operatorname{Spec} \mathbb{F}_q \to U_0$

$$\leadsto Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q) \to \pi_1^{et}(U_0, \bar{x})$$
$$F_x \mapsto$$

$[F_x] \in \pi_1^{et}(U_0, \bar{x}) \xrightarrow{\sim} \pi_1^{et}(U_0, u)$

<u>RHS</u>: $Tr(F \mid H^i_c(U, \mathcal{E})) = Tr(F^{-1} \mid H^{2-i}(U, \mathcal{E}^\vee(1)))$

If U_0 is affine, $F^{-1} \curvearrowright H^{2-i}(U, \mathcal{E}^\vee(1))$

can be interpreted gp-theoretically

b/c $H^{2-i}(U, \mathcal{E}^\vee(1)) \simeq H^{2-i}_{cts}(\pi_1^{et}(U,u), (\mathcal{E}^\vee(1))_u)$

b/c curves are étale π_1-s.

Action of F can be described gp-theoretically as follows:

$$\begin{matrix} U_0 & & 1 \to \pi_1(U,u) \to \pi_1^{et}(U_0,u) \\ \downarrow & \rightsquigarrow & \downarrow \\ \operatorname{Spec} \mathbb{F}_q & & \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \\ & & \downarrow \\ & & 1 \end{matrix}$$

<u>Claim</u> Outer action of

$\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \curvearrowright \pi_1^{et}(U,u)$

induces on $H^i(\pi_1^{et}(U,u), \mathcal{E}^\vee(1)_u)$

action agrees w/ geom. described Frobenius action.

<u>Target thm</u>: U_0 sm. curve$/\mathbb{F}_q$, $\mathcal{E}_0$ locally constant $\mathbb{Q}_\ell$-sheaf on U_0. Then

$$\sum_{x \in U^F} \operatorname{Tr}(F_x | \mathcal{E}_x) = \sum (-1)^r \operatorname{Tr}(F | H^r_c(U, \mathcal{E}))$$

<u>Idea</u> • Formulation for lcc torsion sheaves

• Pass to a cover to reduce to constant sheaves

- Pass to inverse limit to give statement for $\mathbb{Q}_\ell$-sheaves.

<u>Target Thm'</u>
U_0 sm. geom. conn'd curve$/\mathbb{F}_q$, $\mathcal{E}_0$ lcc sheaf of <u>flat</u> $\mathbb{Z}/\ell^n\mathbb{Z}$-modules ($\ell \neq$ char $\mathbb{F}_q$)

Then
$$\sum_{x \in U^F} \mathrm{Tr}(F_x | \mathcal{E}_x) = \sum (-1)^r \mathrm{Tr}(F | H_c^r(U, \mathcal{E}))$$

what does this mean?

<u>Defn</u> (Perfect complex) R - Noetherian local ring
A complex of $R^\bullet$ modules is perfect if it is (quasi-isomorphic to) a bdd complex of f.g. free R-modules.

<u>Prop</u> R-Noeth. local ring $M^\bullet$ complex of R-modules,
$\gamma : P^\bullet \to M^\bullet$ q.i. w/ $P^\bullet$ bdd complex of f.g. free R-modules
$\alpha : M^\bullet \to M^\bullet$ endomorphism. Then $\exists \beta : P^\bullet \to P^\bullet$ s.t.

$$\begin{array}{ccc} P^\bullet & \xrightarrow{\beta} & P^\bullet \\ \gamma\downarrow & & \downarrow\gamma \\ M^\bullet & \xrightarrow{\alpha} & M^\bullet \end{array}$$

commutes, β well-defined up to htpy, and

$$\mathrm{Tr}(\beta \mid P^\bullet) = \sum (-1)^r \,\mathrm{Tr}(\beta \mid P^r)$$

independent of all choices

Pf Non-trivial homological algebra.

Moral reason:

$$\begin{array}{ccccccccc} 0 \to & A & \to & B & \to & C & \to 0 \\ & \downarrow \beta & & \downarrow \beta & & \downarrow \beta & \\ 0 \to & A & \to & B & \to & C & \to 0 \end{array}$$

A, B, C - f.g. free R-modules

Then $\mathrm{Tr}(\beta \mid A) + \mathrm{Tr}(\beta \mid C) = \mathrm{Tr}(\beta \mid B)$.

Rem Not all complexes are perfect!

Non-ex: $\mathbb{Z}/\ell\mathbb{Z}$ as a $\mathbb{Z}/\ell^2\mathbb{Z}$-module

Claim Complex of $\mathbb{Z}/\ell^2\mathbb{Z}$ modules $\mathbb{Z}/\ell\mathbb{Z}[0]$ is not q.i. to a bdd cpx of f.g. free modules.

Pf Suppose it was: $P^\bullet \longrightarrow \mathbb{Z}/\ell\mathbb{Z}[0]$

$\Rightarrow \mathrm{Tor}_i^{\mathbb{Z}/\ell^2\mathbb{Z}}(\mathbb{Z}/\ell\mathbb{Z}, -)$ vanish for $i >> 0$.

But

$$\cdots \to \mathbb{Z}/\ell^2\mathbb{Z} \to \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell^2 \xrightarrow{\cdot \ell} \mathbb{Z}/\ell^2\mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z} \to 0$$

$\Rightarrow \mathrm{Tor}_i^{\mathbb{Z}/\ell^2\mathbb{Z}}(\mathbb{Z}/\ell, \mathbb{Z}/\ell) = \mathbb{Z}/\ell$ for all i. $\lightning$

Criterion for when a complex is perfect:

Prop (Mumford) R Noeth. local ring, $M^\bullet$ cpx of R-modules

- If $H^r(M^\bullet)$ f.g. $\forall r$, $H^r(M^\bullet)=0$ for $r > m$

  $\exists\ Q^\bullet \to M^\bullet$ q.i. w/ Q^i f.g. free, s.t. $Q^r = 0$ for $r > m$.

- If in addition $H^r(Q^\bullet \otimes_R N) = 0$ for $r<0$ and all f.g. R-modules N $\exists$ q.i.

  $Q^\bullet \to P^\bullet$ w/ $P^\bullet$ complex of f.g. free modules supported in $[0, m]$.

tor-amplitude $[0,m]$

Pf of second bullet pt: Replace Q_0 w/ $Q_0/\mathrm{im}\, Q_{-1}$, check it's flat. $\blacksquare$

Obs Proj. formula + finiteness thm $\Rightarrow$ $\mathcal{E}$ flat loc $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaf

$R\Gamma(U_{\acute{e}t}, \mathcal{E})$ satisfies the condition of thm.

$\Longrightarrow R\Gamma(U_{\acute{e}t}, \mathcal{E})$ is perfect.

$$\Longrightarrow \text{“} \sum (-1)^r \mathrm{Tr}(F \mid H^i(U, \mathcal{E})) \text{”}$$

makes sense — b/c can compute it on a q.i. complex which is bdd and consists of f.g. free modules.

Introduction to main geom. ideas of Weil II.

Given $X \subseteq \mathbb{P}^n$ sm. proj. variety

$$\begin{tikzcd} X^\circ \arrow[r] \arrow[d, "\pi_{sm}" ', "\text{fiber dim } d"] & \mathrm{Bl}\, X \arrow[d, "\pi"] \\ U \arrow[r, phantom, "\subseteq"] & \mathbb{P}^1 \end{tikzcd}$$

s.t. fibers have mild singularities & s.t. "monodromy" on middle coh. of fibers has large image.

$$\mathcal{E} = R^d \pi_{sm*} \mathbb{Q}_\ell \in Sh^{\mathbb{Q}_\ell}(U)$$

Idea: List properties of this sheaf and analyze consequences for $H^i(U, \mathcal{E})$.

Allows us to do induction on dim