# Étale Cohomology - 11/17/2020

Last time: Rationality for sm. proj. varieties

$\cdot$ PD $\Rightarrow$ ftnl equ.

Today: RH

Digression on Frobenii:

$X_0$ - variety $/\mathbb{F}_q$

$X = (X_0)_{\overline{\mathbb{F}_q}}$

Absolute Frobenius:

$$\mathrm{Frob}_{abs}: X \longrightarrow X \qquad \leftarrow \text{not a morphism} /\mathbb{F}_q$$
$$f^q \longleftarrow f$$

**Q** $\mathrm{Frob}_{abs}^*: H^*(X, \mathbb{Q}_\ell) \to H^*(X, \mathrm{Frob}_{abs}^* \mathbb{Q}_\ell)$

$$\| $$

$H^*(X, \mathbb{Q}_\ell)$

$H(X, \mathbb{Q}_\ell)$

What map is this?

<u>Claim</u> $\mathrm{Frob}^*_{abs} = \mathrm{id}$

<u>Pf</u> Subclaim:

$\mathrm{Frob}_{abs} : X_{ét} \to X_{ét}$ is nat. isom. to the identity.

$$\begin{tikzcd}
U \arrow[dr, dashed, "F_{U/X}", "\sim"'] \arrow[rr, bend left, "\mathrm{Frob}_{abs}"] \arrow[ddr, bend right, "\pi"'] & & \\
& U \times_{X, \mathrm{Frob}} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "\pi", "ét"'] \\
& X \arrow[r, "\mathrm{Frob}_{abs}"] & X
\end{tikzcd}$$

Exercise:

$F_{U/X}$ isom.

Want: $U \times_X X \simeq U$.

Relative Frobenius:

$$\begin{tikzcd}
X \arrow[dr, dashed, "F_{X/k}"] \arrow[rr, bend left, "\mathrm{Frob}^q_{abs}"] \arrow[ddr, bend right] & & \\
& X \simeq X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
& \mathrm{Spec}\, \overline{\mathbb{F}}_q \arrow[r, "\mathrm{Frob}^q"] & \mathrm{Spec}\, \overline{\mathbb{F}}_q
\end{tikzcd}$$

Concrete description: $X_0 / \mathbb{F}_q$

$\mathrm{Frob}^{q}_{abs, X_0} : X_0 \to X_0 \,/\, \mathbb{F}_q$

$F_{X/k} = \mathrm{Frob}^{q}_{abs, X_0} \times \overline{\mathbb{F}}_q$

<u>Ex</u> $X = \mathbb{A}^n_{\overline{\mathbb{F}}_q} = \mathrm{Spec}\, \overline{\mathbb{F}}_q[t_1, \dots, t_n]$

$\mathrm{Frob}_{abs} : f \mapsto f^p$

$F_{\mathbb{A}^n/\overline{\mathbb{F}}_q} : t_i \longmapsto t_i^p$

Arithmetic/geometric Frobenius:

$F_k = \mathrm{Frob}_{abs} : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$

raises coeffs to $p^{th}$ powers $\rightsquigarrow$ $F_k \times X \; : X \longrightarrow X$

take $p^{th}$ roots of coeffs $\rightsquigarrow$ $F_k^{-1} \times X : X \longrightarrow X$

<u>Obs</u> $\mathrm{Frob}_{abs, X} = (F_k \times id_X) \circ F_{X/k}$

$\Rightarrow F_{X/k}^{*}, \; (F_k \times id_X)^{*}$ act as

inverses to each other on

$H^*(X, \mathbb{Q}_\ell)$.

From now on: Frobenius will always mean $F_{X/k}$.

Thm (RH). $X_0$ sm. proj. variety $/\mathbb{F}_q$, $X = X_{0, \overline{\mathbb{F}}_q}$. Then the eigenvalues

$$F_{X/k}^* \circlearrowright H^i(X, \mathbb{Q}_\ell)$$

are algebraic integers, and for any embedding $\mathbb{Q}(\underset{\alpha_j^i}{\text{eigenvalues}}) \hookrightarrow \mathbb{C}$,

$$|\alpha_j| = q^{i/2}.$$

Ex Curves — Hartshorne IV

Ex If $H^{2i}(X, \mathbb{Q}_\ell(i))$ is spanned by cycle classes, then we know RH for $H^{2i}(X, \mathbb{Q}_\ell)$.

Pf Pick a basis of $H^{2i}(X, \mathbb{Q}_\ell(i))$ consisting of cycle classes. WLOG can assume cycles defined $/\mathbb{F}_q$.

$$\Rightarrow CH^i(X)_{\mathbb{Q}_\ell} \twoheadrightarrow H^{2i}(X, \mathbb{Q}_\ell(i))$$

$F_{X/k}$ equvt.

$$\Rightarrow F_{X/k} \curvearrowright H^{2i}(X, \mathbb{Q}_\ell(i))$$

trivially

$$\Rightarrow H^{2i}(X, \mathbb{Q}_\ell) = H^{2i}(X, \mathbb{Q}_\ell(i))(-i)$$

$F_{X/k} \curvearrowright H^{2i}(X, \mathbb{Q}_\ell)$ via $\chi_{cyc}^{\otimes -i}$.

$\Rightarrow$ eigenvalues $q^i = q^{2i/2}$.

<u>Ex</u> Cubic surface $\Rightarrow H^*$ spanned by cycle classes.

Reductions: (i) Replace $\mathbb{F}_q$ w/ $\mathbb{F}_{q^m}$.
b/c has the effect of replacing $F_{X/k}$ w/ $F_{X/k}^m$, which raises eigenvalues $\alpha_j$ to $m^{th}$ power. But $\alpha_j$ satisfies desired properties $\Leftrightarrow \alpha_j^m$ does (replacing $q$ w/ $q^m$.)

<u>Rem</u> Property that $|\alpha|$ is independent of embedding $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ is very special.

Non-ex: $1+\sqrt{2}$, $1-\sqrt{2}$

<u>Defn</u> If $|\alpha|$ is independent of embedding, $|\alpha| = q^{i/2}$, we say that $\alpha$ is a $q$-Weil number of weight $i$.

Reduction 2: Enough to show:

<u>Thm</u> $X$ sm. proj. of even dim'n $2d$, $\alpha$ eigenvalue of $F_{X/k}$ on $H^{2d}(X, \mathbb{Q}_\ell)$, then $q^{d/2 - \frac{1}{2}} < |\alpha| < q^{d/2 + \frac{1}{2}}$ for all embeddings $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$.

Pf ($\Rightarrow$ RH).

Idea: Take products ("Tensor product trick")

$Y$ - sm. proj. variety of dim'n $n$

dim $= 2nm \rightarrow Y^{2m}$

$$H^{2nm}(Y, \mathbb{Q}_\ell) = \bigoplus_{\substack{i_1, \dots, i_{2m} \\ \sum i_j = 2nm}} \bigotimes_j H^{i_j}(Y, \mathbb{Q}_\ell)$$

(Künneth formula)

• Special case: $i_j = n$

$$\left(H^n(Y, \mathbb{Q}_\ell)\right)^{\otimes 2m} \subseteq H^{2nm}(Y^{2m}, \mathbb{Q}_\ell)$$

middle coh. of even-dim'l variety

$$q^{nm - \frac{1}{2}} < |\alpha^{2m}| < q^{nm + \frac{1}{2}}$$

$$\Rightarrow q^{\frac{n}{2} - \frac{1}{4m}} < |\alpha| < q^{\frac{n}{2} + \frac{1}{4m}}$$

$$\overset{\lim_{m \to \infty}}{\Longrightarrow} |\alpha| = q^{n/2}.$$

$\Rightarrow$ middle coh. of $Y$.

$\overset{PD}{\Longrightarrow}$ enough to do $H^r(Y, \mathbb{Q}_\ell)$ for $r > \dim Y$

$$H^r(Y, \mathbb{Q}_\ell)^{\otimes A} \otimes H^0(Y, \mathbb{Q}_\ell)^{\otimes B} \leq$$

$$H^{Ar+B}(Y^{A+B}, \mathbb{Q}_\ell)$$

Choosing $A, B$ appropriately, can arrange that this is middle cohomology. Repeat the same argument, $A \to \infty$

$\blacksquare$

Need generalization of Lefschetz fixed pt formula

- Non-proper varieties
- Non-constant sheaves
- Sheaves of modules / $\mathbb{Z}/\ell^n\mathbb{Z}$

Non-proper varieties:

$$\underset{}{U} \xrightarrow[\text{open}]{j} \underbrace{X}_{\text{sm. proper variety}} \xhookleftarrow{i} Z = X \setminus U$$

$$0 \to j_! \underline{\mathbb{Q}_\ell}_U \to \underline{\mathbb{Q}_\ell}_X \to i_* \mathbb{Q}_{\ell Z} \to 0$$

$$0 \to H^0_c(U, \mathbb{Q}_\ell) \to H^0(X, \mathbb{Q}_\ell) \to H^0(Z, \mathbb{Q}_\ell) \to H^1_c(U, \mathbb{Q}_\ell) \to H^1(X, \mathbb{Q}_\ell) \to \cdots$$

$\varphi \circlearrowright X$ s.t. $\varphi(U) \subseteq U$, $\varphi(Z) = Z$.

$$\sum (-1)^r \mathrm{Tr}(\varphi^* \mid H^r(X, \mathbb{Q}_\ell)) = \sum (-1)^r \mathrm{Tr}(\varphi^* \mid H^r_c(U, \mathbb{Q}_\ell)) + \sum (-1)^r \mathrm{Tr}(\varphi^* \mid H^r(Z, \mathbb{Q}_\ell))$$

If $Z$ sm. proper, get

$$\sum (-1)^r \mathrm{Tr}(\varphi^* \mid H^r_c(U, \mathbb{Q}_\ell)) = \# X^\varphi - \# Z^\varphi$$

counted w/ multiplicity

$\overset{?}{=} \# U^\varphi$ counted w/ multiplicity

not in general

**Ex** $X = \mathbb{P}^1$, $U = \mathbb{A}^1$, $Z = \infty$

$\varphi : [x_0 : x_1] \longmapsto [x_0 + x_1 : x_1]$

$\# U^\varphi = 0 \qquad \# \mathbb{P}^{1\varphi} = 2$

$\# Z^\varphi = 1$

$2 - 1 \neq 0$

Computing mult. on $\mathbb{P}^1$ is not the same as computing on $\infty$.

Get: Lefschetz fixed pt formula

$\overset{?}{\Longrightarrow} \# U^\varphi + \# Z^\varphi = \# X^\varphi$

$\overset{?}{\Longrightarrow} \forall x \in Z^\varphi$, $\mathrm{mult}_\varphi(x, Z) = \mathrm{mult}_\varphi(x, X)$.

**Ex** If $\varphi$ = Frobenius, this always ok b/c all fixed pts have multiplicity 1.

**Cor** $U$ sm. w/ sm. cpctfication + sm complement,

$$\#U(\mathbb{F}_{q^n}) = \sum (-1)^r \operatorname{Tr}(F_{U/\mathbb{F}_q}^{n} \mid H_c^r(U, \mathbb{Q}_\ell))$$

**Cor** $U_0$ variety$/\mathbb{F}_q$

$$\#U_0(\mathbb{F}_{q^m}) = \sum (-1)^r \operatorname{Tr}(F_{U/\mathbb{F}_q}^{m} \mid H_c^r(U, \mathbb{Q}_\ell))$$

Non-constant sheaves:

$\mathcal{E}$ - lisse $\mathbb{Q}_\ell$-sheaf on $X$

$\varphi : X \to X$ , $H^*(X, \mathcal{E}) \to H^*(X, \varphi^*\mathcal{E})$

In order to take traces, need in addition a map $\varphi_{\mathcal{E}} : \varphi^*\mathcal{E} \to \mathcal{E}$.

$$(\varphi, \varphi_{\mathcal{E}})^* : H^*(X, \mathcal{E}) \to H^*(X, \varphi^*\mathcal{E}) \xrightarrow{\varphi_{\mathcal{E}}} H^*(X, \mathcal{E})$$

If $x \in X^{\varphi}$ is a fixed pt

$$\begin{tikzcd}
(\varphi^*\mathcal{E})_x \arrow[r, "\varphi_\mathcal{E}"] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{E}_x \\
(\mathcal{E})_{\varphi(x)} \arrow[d, phantom, "\cong" rotate=90] & \\
\mathcal{E}_x \arrow[ruu, bend right, "\varphi_x"'] &
\end{tikzcd}$$

Q Does

$$\sum_{x\in X^\varphi} Tr(\varphi_x \mid \mathcal{E}_x) = \sum_r (-1)^r Tr((\varphi,\varphi_\mathcal{E})^* \mid H^r_c(X,\mathcal{E}))$$

Ex $X$ - finite set.

$\varphi: X \to X$ $\qquad$ $\mathcal{E}_x$ - $\mathbb{Q}_\ell$ v.s. for each $x$

$$\varphi_\mathcal{E}: \mathcal{E}_{\varphi(x)} \xrightarrow{\varphi_x} \mathcal{E}_x \quad \forall x.$$

$$\sum_{x\in X^\varphi} Tr(\varphi_x) = Tr\Big(\bigoplus \varphi_x : \bigoplus \mathcal{E}_x \to \bigoplus \mathcal{E}_x\Big)$$

Pf Only diagonal blocks

contribute to these.