Étale Cohomology - 11/12/2020

Last time: Lefschetz fixed point formula

X - sm. proj. variety$/_{k=\bar{k}}$

$\varphi: X \to X$

$$\Gamma_\varphi \cdot \Delta = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(\varphi \mid H^i(X, \mathbb{Q}_\ell))$$

<u>Ex</u> $X = \bigsqcup_{i=0}^{n} \operatorname{Spec} k$

$\varphi: X \to X$

$\#$ fixed pts of $\varphi = \operatorname{Tr}(\varphi \mid \mathbb{Q}_\ell^X)$

<u>Ex</u> $f: \mathbb{P}^1 \to \mathbb{P}^1$ deg d

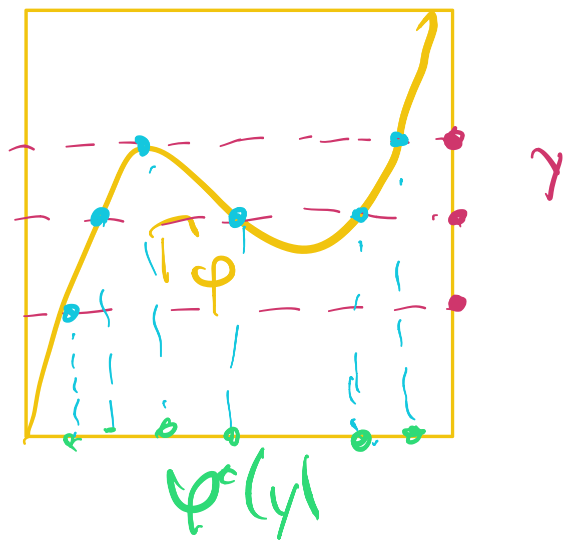
Preparations:

<u>Lemma 1</u> Given $\varphi: X \to Y$ (X, Y sm. proj)

$y \in H^*(Y, \mathbb{Q}_\ell)$

$$\varphi^*(y) = p_*(cl(\Gamma_\varphi) \cup q^*(y))$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$



Pf. Exercise

Lemma 2. e_i^r basis of $H^r(X, \mathbb{Q}_\ell)$

$f_i^{2\dim X - r}$ dual basis of $H^{2\dim X - r}(X, \mathbb{Q}_\ell)$

$$cl(\Gamma_\varphi) = \sum \varphi^*(e_i) \otimes f_i$$

(Where we view $e_i \otimes f_j$ as a basis for $H^*(X \times X, \mathbb{Q}_\ell)$ via Künneth).

Ex $\varphi = id$, $cl(\Gamma_\varphi) = cl(\Delta) = \sum_i e_i^r \otimes f_i^{2\dim X - r}$

Pf $cl(\Gamma_\varphi) = \sum a_i \otimes f_i$ for unique $a_i \in H^*(X, \mathbb{Q}_\ell)$

b/c $H^*(X \times X, \mathbb{Q}_\ell)$ is free as a right $H^*(X, \mathbb{Q}_\ell)$-module.

Goal: $a_i = \varphi^*(e_i)$

$$\varphi^*(e_i) = p_*\left(cl(\Gamma_\varphi) \cup q^* e_i\right)$$
$$= p_*\left(\sum_j (a_j \otimes f_j) \cup q^* e_i\right)$$
$$= p_*\left(\sum_j \left((a_j \otimes f_j) \cup (1 \otimes e_i)\right)\right)$$
$$= p_*\left(\sum_j a_j \otimes (f_j \cup e_i)\right)$$
$$= p_*\left(a_i \otimes e_i^{2\dim X}\right)$$
$$= a_i. \quad \square$$

Pf Sketch (of Lefschetz)

(Assume that cycle class map

(Assume
sends intersection product
to cup product.)

$$cl(\Gamma_\varphi) = \sum \varphi^* e_i \otimes f_i$$

$$cl(\Delta) = \sum e_i \otimes f_i$$

$$cl(\Gamma_\varphi \cdot \Delta) = cl(\Gamma_\varphi) \cup cl(\Delta)$$

$$= \left(\sum \varphi^* e_i \otimes f_i\right) \cup \left(\sum f_i \otimes e_i\right)$$

$$= \sum_{i,j} (\varphi^* e_i) f_j \otimes f_i e_j$$

$f_i e_j$: $e_{2\dim X}$ if $i = j$, 0 otherwise

$$= \sum_i (\varphi^* e_i) f_i \otimes e_{2\dim X}$$

Want: $\mathrm{Tr}\left(\sum (\varphi^* e_i) f_i\right) = \mathrm{Tr}(\varphi \mid H^*)$

(up to sign issue)

Claim: e_i basis of V, $e_i^\vee$ dual basis

$$\mathrm{Tr}(A) = \sum_i e_i^\vee(A e_i). \checkmark$$

∎

Q What is $\Gamma_\varphi \cdot \Delta$?

Claim $\Gamma_\varphi \cdot \Delta = \#$ fixed pts of φ
(counted w/out multiplicity)
if multiplicity of each fixed pt is 1.

Claim $Y, Z \subseteq$ sm. X are subvarieties
of complementary dim'n.

Then $(Y \cdot Z)_p = 1$ if

- Y, Z sm. at p.
- $T_p Y \cap T_p Z = 0$.

Lemma. Satisfied for Γ_φ, Δ if
X smooth, 1 is not an eigenvalue
of φ-action on $T_p X$ for p any
fixed pt of φ.

Ex Hypotheses of Lemma true for
$X / \mathbb{F}_p$

$\varphi : X \to X$ absolute Frobenius map.

Interlude on Frobenius maps:

X variety$/\mathbb{F}_q$

$$\begin{tikzcd} X \arrow[rr, "\mathrm{Frob}^q"] \arrow[dr] & & X \arrow[dl] \\ & \operatorname{Spec} \mathbb{F}_q & \end{tikzcd} \qquad (f^q \mapsfrom f)$$

$$\begin{tikzcd} X_{\overline{\mathbb{F}}_q} \arrow[r, "F^q_{X_{\overline{\mathbb{F}}_q}/\overline{\mathbb{F}}_q}"] \arrow[rr, bend left, "\mathrm{Frob}^q"] \arrow[dr, bend right] & X_{\overline{\mathbb{F}}_q} \cong X'_{\overline{\mathbb{F}}_q} \arrow[r] \arrow[d] & X_{\overline{\mathbb{F}}_q} \arrow[d] \\ & \operatorname{Spec} \overline{\mathbb{F}}_q \arrow[r, "\mathrm{Frob}^q"] & \operatorname{Spec} \overline{\mathbb{F}}_q \end{tikzcd}$$

identity on topological spaces.

In affine setting: $X = \operatorname{Spec} \mathbb{F}_q[x_1, \dots, x_n]/I$

$X_{\overline{\mathbb{F}}_q} = \operatorname{Spec} \overline{\mathbb{F}}_q[x_1, \dots x_n]/I$

$X'_{\overline{\mathbb{F}}_q} = \operatorname{Spec} \overline{\mathbb{F}}_q \otimes_{\overline{\mathbb{F}}_q} \overline{\mathbb{F}}_q[x_1 \dots x_n]/I$

Claim: Fixed pts of $F^q_{X_{\overline{\mathbb{F}}_q}/\overline{\mathbb{F}}_q} = F^q$ are the $\mathbb{F}_q$-

rat'l pts of X.

- All fixed pts have multiplicity 1.

<u>Pf of 2</u> Enough to show

F^q induces the zero on tangent spaces.

$$\operatorname{Spec} \mathbb{F}_q[t]/t^2 \to X \to X \quad \boxed{\checkmark}$$

Proof of Weil Conjectures:

(1) X variety$/\mathbb{F}_q$. Want:

$$\zeta_X(t) = \exp\left(\sum_{i=1}^{\infty} \frac{\#X(\mathbb{F}_{q^i})}{i} t^i\right)$$

is rat'l fn of t.

<u>Pf for X sm. proj.</u>

<u>Defn</u> $\operatorname{Tr}(F^q \mid H^*(X)) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(F^q \mid H^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell))$

$$\# X(\mathbb{F}_{q^i}) = \mathrm{Tr}(F^i \mid H^*(X))$$
$$= \mathrm{Tr}((F^q)^i \mid H^*(X))$$

$$S_X(t) = \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{Tr}((F^q)^n \mid H^*(X))}{n} t^n\right)$$

ETS: $S_{X,r} = \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{Tr}((F^q)^n \mid H^r(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) t^n}{n}\right)$

is rat'l.

b/c $S_X(t)$ alternating product of those as $r = 0 \ldots 2\dim X$.

<u>WLOG</u> $\mathrm{Tr}((F_q)^n \mid H^r(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = \sum_{i=1}^{\dim H^r} \lambda_i^n$

f.d. vector space

$$S_{X,r} = \exp\left(\sum_{n=1}^{\infty} \sum_{i=1}^{\dim H^r} \frac{\lambda_i^n t^n}{n}\right)$$
$$= \exp\left(\sum_{i=1}^{\dim H^r} -\log(1 - \lambda_i t)\right)$$

$$= \prod \frac{1}{(1-\lambda_i t)} \in \mathbb{Q}_\ell(t)$$ ← rat'l ftn!

$\exists$

ζ_X = Rat'l ftn w/ integer coeffs:

$$d\log \zeta_X(t) = \sum \#X(\mathbb{F}_{q^n})\, t^{n-1}$$

integer coeffs.

Functional Equation:

Poincaré duality:

$$H^i(X, \mathbb{Q}_\ell) \text{ dual to } H^{2\dim X - i}(X, \mathbb{Q}_\ell(d))$$

$$\zeta_{X,r}(t) \longleftrightarrow \zeta_{X, 2\dim X - i}(q^d t)$$

<u>RH</u>: (X sm. proj.)

<u>Thm</u> $F^{2} \curvearrowright H^r(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ has eigenvalues which are algebraic #'s, & for any embedding

$\mathbb{Q}(\lambda_1, \dots, \lambda_{\dim H^r}) \hookrightarrow \mathbb{C}$, absolute values of λ_i all equal to $q^{r/2}$.

Ex Know eigenvalues are alg. #'s e.g. for hypersurfaces.

$X \subseteq \mathbb{P}^n$ sm. proj. hypersurface

$H^*(X, \mathbb{Q}_\ell) \xrightarrow{\text{Lefschetz hyperplane thm.}} H^*(\mathbb{P}^n, \mathbb{Q}_\ell)$ $* < \dim X$

b/c $\mathbb{P}^n \setminus X$ affine. + excision.

+ Poincaré duality

PD $\Rightarrow$ Know $H^*(X, \mathbb{Q}_\ell)$ is a twist of $\mathbb{Q}_\ell$ unless $* = \dim X$.