Étale Cohomology – 11/10/2020

Last time: X sm variety$/k_{=k^s}$ of dim'n d

$$H^{2d}_c(X, \Lambda(d)) \xrightarrow[\sim]{Tr} \Lambda$$

$\underline{\mathbb{Z}/\ell^n\mathbb{Z}}$ or $\underline{\mathbb{Z}_\ell}$ or $\underline{\mathbb{Q}_\ell}$

Today: · Sketch use of this for Poincaré duality.
· Lefschetz fixed point formula

Recall statement:

$$\begin{array}{c} H^i_c(X, \Lambda(d)) \times H^{2d-i}(X, \Lambda) \\ \downarrow \smile \\ H^{2d}_c(X, \Lambda(d)) \xrightarrow[\sim]{Tr} \Lambda \end{array}$$

If $\Lambda = \underline{\mathbb{Q}_\ell}$, this is a perfect pairing.

<u>Idea</u>: Find a relative version
(Verdier duality)

$f: X \to Y$ morphism of k-varieties

$$Rf_! : D^c(X) \longrightarrow D^c(Y)$$

cat. of bdd complexes of Abelian sheaves constructible cohomology, on étale site, formally invert quasi-iso,

Idea: (1) Construct a right adjoint to $Rf_!$

$$f^! : D^c(Y) \to D^c(X)$$

(2) Compute that if f sm. of pure dim'n d,

$$f^!(\mathcal{F}) = \mathcal{F}(d)[2d]$$

(3) Adjointness $\Rightarrow$

$$Rf_* R\underline{Hom}^{D^c(X)}(\mathcal{F}, f^! \mathcal{G}) \xrightarrow{\sim} R\underline{Hom}^{D^c(Y)}(Rf_! \mathcal{F}, \mathcal{G})$$

What does this have to do w/ Poincaré duality???

X - sm. variety, Y - pt, $\mathcal{F}, \mathcal{G} = \Lambda$ ← const. sheaf

$$\Lambda(d)[2d]$$

$$Rf_* \underline{R\mathrm{Hom}}(\Lambda, f^!\Lambda) \leftarrow Rf_*\Lambda(d)[2d]$$

computes $H^{i-2d}(X, \Lambda(d))$

$$\downarrow \wr$$

$$R\mathrm{Hom}(Rf_!\Lambda, \Lambda)$$

$$\Leftarrow \mathrm{Ext}^i(H_c^j(X,\Lambda), \Lambda)$$

$\Lambda = \mathbb{Q}_\ell$: $\quad H_c^{-j}(X,\Lambda)^\vee \simeq H^{j-2d}(X, \Lambda(d))$

← Poincaré duality.

$\Lambda = \mathbb{Z}_\ell$:

$$\mathrm{Ext}^i(H_c^j(X,\Lambda),\Lambda)$$

$$\Downarrow$$

$$H^{i-j+2d}(X, \Lambda(d))$$

Modern construction of $f^!$ (due to Neeman): Use Brown representability

(1) What is $\mathrm{Hom}(-, f^!\mathcal{G})$

$\mathrm{Hom}(Rf_!-, \mathcal{G})$

Checks that this is rep'ble.

(2) Compute it for f smooth.

(i) Reduce to

$f: X \to S$ sm. of rel. dim'n $\underline{1}$

$\mathcal{F} = \Lambda$

$\mathcal{G}$ - constructible

"direct computation" — one simple trick if S is a pt, reduce to situation $/\mathbb{C}$.

$$f^!\mathcal{F} \cong f^*\mathcal{F}(d)[2d].$$

<u>Cor</u> X sm. variety of dim'n d, $/k = \bar{k}$

$$H^i(X, \overset{\text{constr}}{\mathcal{F}}) = 0 \quad \text{for } i > 2d.$$

Pf dual to something in negative degrees

Rem True for arbitrary varieties of dim'n d.

Non-ex Curve X/$\mathbb{F}_q$, $H^3_c(X, \Lambda(1)) \neq 0$.

Lefschetz fixed pt formula:

X sm. proj. $/k = k^s$, $\varphi : X \to X$

$\Delta \subseteq X \times X$ $\quad \ell \in$ char k

Thm

$$\deg \Gamma_\varphi \cdot \Delta = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}(\varphi^* | H^i(X_{\acute{e}t}, \mathbb{Q}_\ell))$$

\# fixed pts

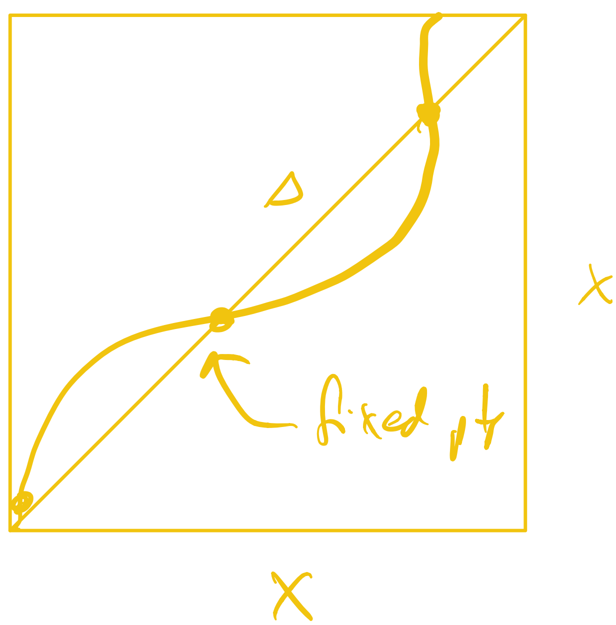
graph of $\varphi \subseteq X \times X$

$\Gamma_\varphi \cdot \Delta$ - intersection of 2 cycles of codim $\dim X$ in $X \times X$

$\deg \Gamma_\varphi \cdot \Delta$ is a number.

In this class: $\operatorname{Tr}(\underbrace{cl(\Gamma_\varphi) \cup cl(\Delta)}))$

$$H^{4\dim(X)}(X\times X, \mathbb{Q}_\ell(2\dim(X)))$$
$$\downarrow \mathrm{Tr}$$
$$\mathbb{Q}_\ell$$



<u>Cor</u> $\deg \Delta\cdot\Delta = \chi(H^i(X,\mathbb{Q}_\ell))$

<u>Rem</u> We'll apply this when $\varphi = \mathrm{Frob}$.

Gysin maps: $\pi: Y\to X$ proper

X, Y smooth varieties, proper

$\pi_*: H^r(Y,\Lambda)\to H^{r-2c}(X,\Lambda(-c))$

where c is the relative dim of π.

$c = \dim(Y) - \dim(X)$.

(dual to π^*).

Properties: (1) Defining property:

$y \in H^r(Y)$, $x \in H^{2\dim Y - r}(X)$

$$\mathrm{Tr}_X(\pi_*(x) \cup y) = \mathrm{Tr}_Y(x \cup \pi^*(y))$$

(2) π is a closed immersion:

$$\pi_*(1) = cl(Y)$$

(3) $(\pi_1 \circ \pi_2)_* = (\pi_1)_* \circ (\pi_2)_*$

(4) $\pi_*(y \cup \pi^*(x)) = \pi_*(y) \cup x$

(projection formula)

(5) π finite of degree d

$$\pi_* \circ \pi^* = d \cdot \mathrm{id}.$$

In general (if X, Y not proper)

$\pi_* : H^v_c(Y, \Lambda) \to H^{v-2c}_c(X, \Lambda(-c))$

X sm. not proper: Lefschetz formula w/ compactly supported cohomology.

<u>Ex</u> (Lefschetz fixed pt formula)

$X = \mathbb{P}^1/_{k=\bar{k}}$ char $k = 0$.

$\varphi : x \mapsto x^n$ $\qquad x = x^n \quad \{0, \mu_{n-1}, \infty\}$

$\# \Gamma_\varphi \cap \Delta = n+1 \qquad x^n - x = 0$

$$\sum_{i=0}^{2} (-1)^i \operatorname{Tr}(\varphi \mid H^i(\mathbb{P}^1, \mathbb{Q}_\ell)) =$$

$$\underbrace{\operatorname{Tr}(\varphi^* \mid H^0(\mathbb{P}^1, \mathbb{Q}_\ell))}_{1} + \underbrace{\operatorname{Tr}(\varphi^* \mid H^2(\mathbb{P}^1, \mathbb{Q}_\ell))}_{n}$$

$H^0(\mathbb{P}^1, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \qquad H^2(\mathbb{P}^1, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$

$$H^2(\mathbb{P}^1, \mu_{\ell^n}) = \operatorname{coker}(\operatorname{Pic} \mathbb{P}^1 \xrightarrow{f \mapsto f^{\ell^n}} \operatorname{Pic} \mathbb{P}^1)$$

$$H^2(\mathbb{P}^1, \mathbb{Q}_\ell) = \left(\varprojlim_n \operatorname{coker}(\operatorname{Pic} \mathbb{P}^1 \xrightarrow{\ell^n} \operatorname{Pic} \mathbb{P}^1)\right) \otimes \mathbb{Q}_\ell$$

$\varphi : [n] \cdot \mathrm{id}$ on $\mathbb{P}_{1/2}\,\mathbb{P}^1$, here on $H^2(\mathbb{P}^1, \mathbb{Q}_\ell)$

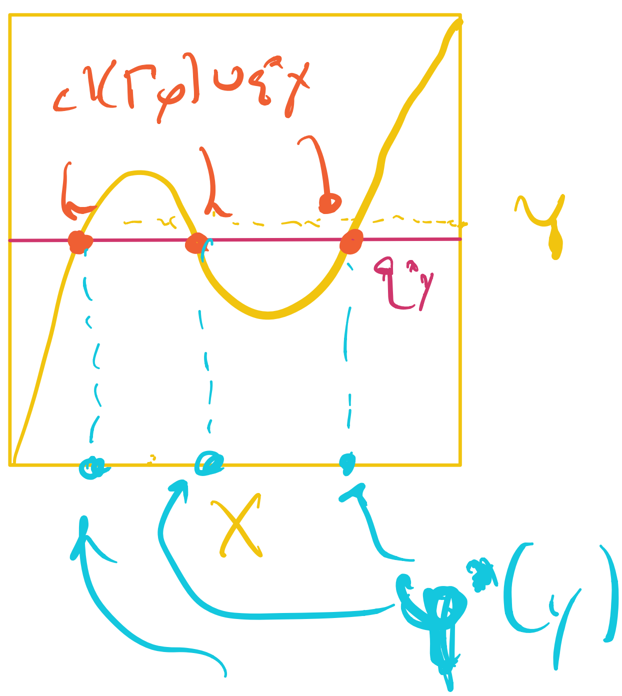
Sketch Pf of Lefschetz fixed pt formula:

<u>Lemma</u>: $\varphi : X \to Y$, $y \in H^*(Y, \mathbb{Q}_\ell)$

fix an iso $\mathbb{Q}_\ell \xrightarrow{\sim} \mathbb{Q}_\ell(1)$

$$\varphi^*(y) = p_*(cl(\Gamma_\varphi) \cup q^* y)$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$



Pf Exercise (look at Milne)
Content: defn of p_*, *
projection formula.

<u>Lemma</u> e_i^r basis of $H^r(X, \mathbb{Q}_\ell)$

f_i^{2d-r} dual basis of $H^{2d-r}(X, \mathbb{Q}_\ell(d))$

Then $cl_{X\times X}(\Gamma_\varphi) = \sum \varphi^*(e_i^r) \otimes f_i^{2d-r}$

under Künneth isomorphism

$$H^*(X\times X, \mathbb{Q}_\ell(d)) \simeq H^*(X, \mathbb{Q}_\ell) \otimes H^*(X, \mathbb{Q}_\ell(d))$$

Pf Next time.