Étale Cohomology - 12/3/2020

Lefschetz fibration:

$X \subseteq \mathbb{P}^n$ sm. proj. variety

$L \subseteq \check{\mathbb{P}}^n$ s.t. (i) Base locus of L intersects X transversely

(ii) $X_t = X \cap H_t$ is smooth for almost all $t \in L$

(iii) For X_t singular, X_t has a unique singular point, ordinary double point

$$\begin{array}{l} \tilde{X} = \mathrm{Bl}_{\text{base locus}} X \\ \downarrow \pi \\ L \end{array} \qquad \pi^{-1}(t) = X_t.$$

Let $S \subseteq L$ be the set of pts s.t. X_t singular

Then $R^i \pi_* \mathbb{Q}_\ell \big|_{L \setminus S}$ locally const $\mathbb{Q}_\ell$-sheaves

$\rightsquigarrow \pi_1^{et}(L \setminus S) \to GL_n(\mathbb{Q}_\ell)$

For each $s \in S$ $\quad \operatorname{Spec}(\operatorname{Frac} \mathcal{O}_{L,s}) \to L \setminus S$

$$\downarrow \otimes \bar{k}$$

$$\pi_1^{\text{ét}}(\operatorname{Spec} \operatorname{Frac}(\widehat{\mathcal{O}_{L_{\bar{k}},s}})) \to \pi_1^{\text{ét}}((L\setminus S)_{\bar{k}})$$

$$\widehat{\mathcal{O}_{L_{\bar{k}},s}} \leftarrow \bar{k}((t))$$

inertia at s $\longrightarrow I_s \subseteq \pi_1^{\text{ét}}(\operatorname{Spec} \operatorname{Frac}(\widehat{\mathcal{O}_{L_{\bar{k}},s}}))$

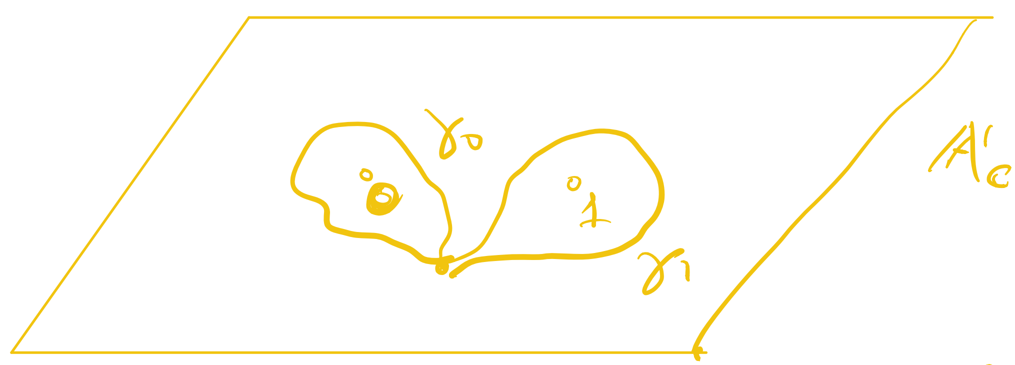
<u>Ex</u> $\{y^2 = x(x-1)(x-\lambda)\} = E$

$$\pi \downarrow$$

$\mathbb{A}^1_\lambda$, smooth over $\mathbb{A}^1_\lambda \setminus \{0,1\}$

Fibers over $0,1$ are nodal cubics

$/\mathbb{C}$: $\pi_1(\mathbb{A}^1_{\mathbb{C}} \setminus \{0,1\}^{an}) = \langle \gamma_0, \gamma_1 \rangle$



$R^1\pi_* \mathbb{Q}_\ell \leftarrow$ lc $\mathbb{Q}_\ell$-sheaf on $\mathbb{A}^1 \setminus \{0,1\}$

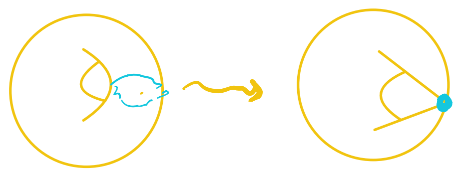
$\pi_1(\mathbb{A}^1 \setminus \{0,1\}^{an}) \to GL_2(\mathbb{Q}_\ell)$

$\gamma_0 \longmapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

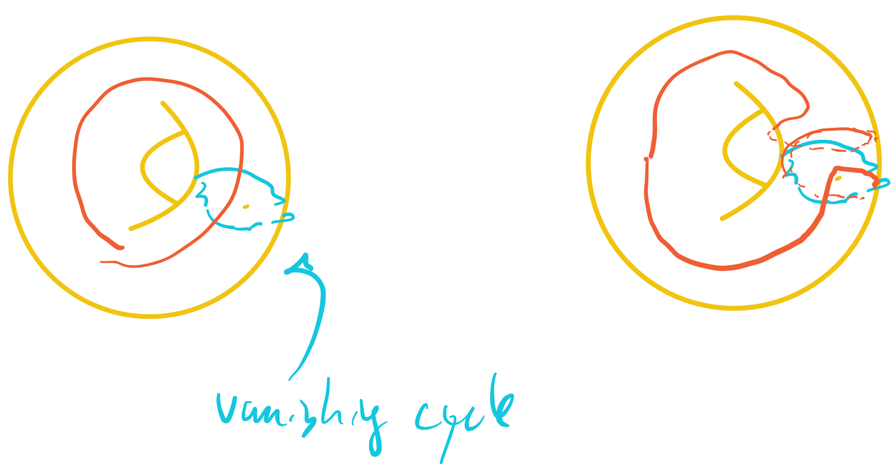
$\gamma_1 \longmapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

topologically generate an open subgroup of $SL_2(\mathbb{Q}_\ell)$.

In a nbd of 0:



$\varepsilon \rightsquigarrow 0$



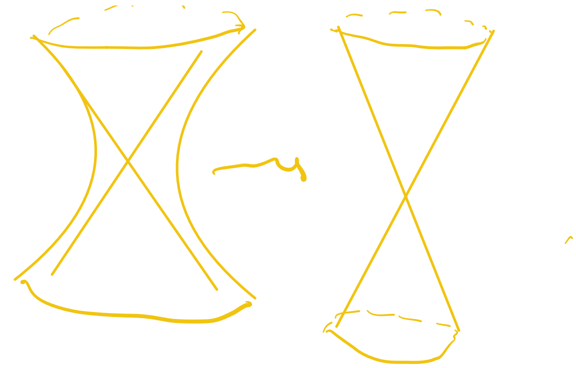
Pairing: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Rem: Properties of these reps depend on parity of coh. degree.

Ex $\{x^2+y^2+z^2+tw^2=0\}$

$$\downarrow$$

$$\mathbb{P}^1_t$$

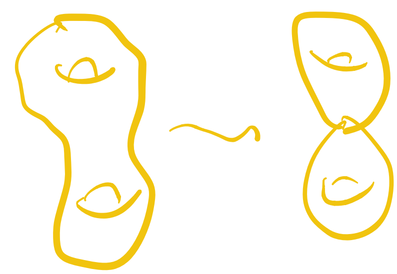


Inertia acts via reflections on H^2.

$\pi_1(\mathbb{A}^1\setminus\{0\}) = \mathbb{Z}$, generator acts on $H^2(X_t, \mathbb{Z})$ via $\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

Claims about Lefschetz fibrations w/ odd-dim'l fibers (fiber dim is $n=2m+1$):

(1) For $r\neq n, n+1$, $R^r\pi_*\mathbb{Q}_\ell$ loc. const. on L.



(2) $R^n\pi_*\mathbb{Q}_\ell|_{L\setminus S}$ loc. const., tame.

(3) For $s\in S$ short exact

$$0\to H^n(X_s,\mathbb{Q}_\ell)\to H^n(X_{\bar\eta},\mathbb{Q}_\ell)\xrightarrow{-\cup\delta_s}\mathbb{Q}_\ell(m-n)\to 0$$

$$\delta_s \in H^n(X_{\bar\eta}, \mathbb{Q}_\ell)^\vee(m-n) = H^n(X_{\bar\eta}, \mathbb{Q}_\ell(n))(m-n)$$
$$= H^n(X_{\bar\eta}, \mathbb{Q}_\ell(m))$$

image of 1 under the natural map

$\mathbb{Q}_\ell \to H^n(X_{\bar\eta}, \mathbb{Q}_\ell(m))$ dual to $\cdot \cup \delta_s$.

Concrete way understand $\mathrm{Span}(\delta_s)$: $\ker\left(H^n(X_{\bar\eta}, \mathbb{Q}_\ell(m))^\vee \to H^n(X_s, \mathbb{Q}_\ell(m))^\vee\right)$

vanishing cycles.

(4) $\sigma_s \in I_s$ acts on $x \in H^n(X_{\bar\eta}, \mathbb{Q}_\ell)$

$\sigma_s(x) = x \pm t(\sigma_s)(x \cup \delta_s)\, \delta_s$

$t : I_s \to \mathbb{Z}_\ell(1)$

(Picard-Lefschetz formula).

<u>Pf of RH</u>: Understand $H^{n+1}(X, \mathbb{Q}_\ell)$

Compute via Leray spectral sequence.

Interesting part: $H^1(\mathbb{P}^1, R^n\pi_* \mathbb{Q}_\ell)$

understand this sheaf better.

Goal: understand π_1 $\mathbb{1}$ Span of vanishing cycles in $H^n(X_{\bar{\eta}}, \mathbb{Q}_\ell(m))$

Let $E \subseteq H^n(X_{\bar{\eta}}, \mathbb{Q}_\ell)$ span of vanishing cycles.

<u>Prop</u> E stable under $\pi_1(L \setminus S)$,

$$E^\perp = H^n(X_{\bar{\eta}}, \mathbb{Q}_\ell)^{\pi_1(L \setminus S)}$$

<u>Pf</u> (1) $\{I_s\}_{s \in S}$ generate $\pi_1^{et}(L/S)^{tame}$

(b/c I_s generate $\pi_1(\mathbb{P}^1 \setminus S)$ in char 0)

(2) ETS $I_s(\delta_{s'}) \subseteq E$

$$\sigma_s(\delta_{s'}) = \delta_{s'} \pm t(\sigma_s)(\delta_{s'} \cup \delta_s)\delta_s \in E.$$

(3) Suppose $x \in E^\perp$. Then

$\sigma_s(x) = x \pm f(\sigma_s)(x \cancel{\cup \delta_s})\,\delta_s$

$\qquad\qquad\qquad\qquad\quad 0$

$\Rightarrow E^{\perp} \subseteq H^1(X_{\bar\eta}, \mathbb{Q}_\ell)^{\pi_1}$.

(other direction is same argument). ☑

Idea: $0 \subseteq E \cap E^{\perp} \subseteq E \subseteq H^n(X_{\bar\eta}, \mathbb{Q}_\ell)$

constant local system

(1) $E \cap E^{\perp}$ constant

(2) $H^n(X_{\bar\eta}, \mathbb{Q}_\ell)/E$ constant (by Picard Lefschetz)

(3) $E/E \cap E^{\perp}$ interesting.

<u>Prop</u> (1) char. of Frobenius on $E/E\cap E^{\perp}$ rat'l

(2) non-deg skew-symmetric pairing $\psi: E/E\cap E^{\perp} \times E/E\cap E^{\perp} \to \mathbb{Q}_\ell(-n)$

(3) $\mathrm{Im}(\pi_1(L \setminus S))$ in $Sp(E/E\cap E^{\perp}, \psi)$ open.

<u>Pf</u> (1) Skip.

(2) Pairing inherited from cup product.

(3) (i) $E/E\cap E^{\perp}$ absolutely simple as a π_1-rep'n

(ii) Thm (Kazhdan-Margulis)

ψ non-degenerate symplectic form on a $\mathbb{Q}_\ell$-v.s.

W, $G \subseteq Sp(W, \psi)$ closed subgp s.t.

(a) W absolutely simple G-module

(b) G generated by transvections
$$x \mapsto x \pm \psi(x, \delta)\delta$$
Then G contains an open subgp of $Sp(W, \psi)$.

Pf idea (Kazhdan-Margulis)

(a) G - ℓ-adic Lie subgp.

(b) ETS $\mathrm{Lie}\, G = \mathrm{Lie}\, Sp(W, \psi)$.

(c) Transvections $\Rightarrow$ $\mathrm{Lie}\, G$ gen'd by
$$x \mapsto \pm \psi(x, \delta)\delta.$$

(d) Linear algebra.

$\Rightarrow$ $E/E \cap E^{\perp}$ satisfies hyp of MAIN LEMMA.