Étale Cohomology - 12/15/2020

Strategy for proof of RH:

Reduced to: X even dim'l of dim'n $n+1$,
want eigenvalues α of Frob $\curvearrowright H^{n+1}(X_{\bar{k}}, \mathbb{Q}_\ell)$
satisfy $q^{n/2} \leq |\alpha| \leq q^{n/2+1}$

Pf by induction on n:

(1) Find an embedding $X \hookrightarrow \mathbb{P}^N$ s.t. there exists Lefschetz pencil L.

(2) ETS theorem for

$$\mathrm{Bl}_{\text{base locus}(L)} X$$

b/c $H^*(X) \hookrightarrow H^*(\mathrm{Bl}\, X)$

so will replace X w/ Bl.

(3) $$\begin{array}{c} \mathrm{Bl}\, X \\ \downarrow \pi \\ \mathbb{P}^1 \end{array}$$ Lefschetz fibration.

Leray s.s.

$$H^i(\mathbb{P}^1, R^j\pi_* \mathbb{Q}_\ell) \Rightarrow H^{i+j}(Bl\, X, \mathbb{Q}_\ell)$$

Interesting grps:

(i) $H^2(\mathbb{P}^1, R^{n-1}\pi_* \mathbb{Q}_\ell)$

(ii) $H^1(\mathbb{P}^1, R^n\pi_* \mathbb{Q}_\ell)$

(iii) $H^0(\mathbb{P}^1, R^{n+1}\pi_* \mathbb{Q}_\ell)$.

(i) $H^2(\mathbb{P}^1, R^{n-1}\pi_* \mathbb{Q}_\ell)$ constant

$H^2(\mathbb{P}^1, \underline{H^{n-1}(\text{fiber})}) = H^{n-1}(\text{fiber})(-1)$

Fiber is a variety of odd dim'n n. Taking a hyperplane section Z

$H^{n-1}(\text{fiber}) \underset{\text{WL}}{\hookrightarrow} H^{n-1}(Z)$ middle coh. of an even dim'l variety ☐

(iii) $H^0(R^{n+1}\pi_* \mathbb{Q}_\ell)$ constant sheaf in good situations

The same argument + PD ⇒ we win.

(ii) $H^1(\mathbb{P}^1, R^n\pi_*\mathbb{Q}_\ell)$

$$R^n\pi_*\mathbb{Q}_\ell \supseteq E \supseteq E\cap E^\perp$$

(ii)(a) $R^n\pi_*\mathbb{Q}_\ell/E$

(b) $E/E\cap E^\perp$ ← satisfies hyp. of MAIN LEMMA
⇒ $H^1(\mathbb{P}^1, E/E\cap E^\perp)$ satisfies inequality

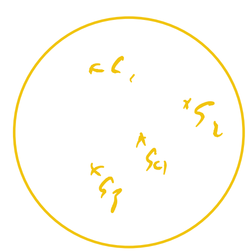
(c) $E\cap E^\perp$

Claim $R^n\pi_*\mathbb{Q}_\ell/E$ and $E\cap E^\perp$ are constant sheaves on $\mathbb{P}^1$.

Pf Picard Lefschetz formula

$R^n\pi_*\mathbb{Q}_\ell$: $\sigma_s(x) = x \pm t(\sigma)(x\cup\delta_s)\delta_s$ ← lives in E.

$E\cap E^\perp$: $x\cup\delta_s = 0 \quad \forall s$



Done by Weak Lefschetz.

Weil II + Applications

Thm (Deligne)

$U/\mathbb{F}_q$ sm. geom. conn'd curve, $\mathcal{F}$ $\mathbb{Q}_\ell$ sheaf on U which is pure of wt zero.

$F_x \curvearrowright \mathcal{F}_x$ has eigenvalues α s.t. $|\alpha| = 1$

Then $H^1_c(U_{\overline{\mathbb{F}}_q}, \mathcal{F})$ is mixed of wts ≤ 1.

eigenvalues α of F satisfy $|\alpha| = q^{i/2}$ where $i \in \mathbb{Z}_{\geq -1}$.

Cor (PD) $H^1(U_{\overline{\mathbb{F}}_q}, \mathcal{F})$ is mixed of wts ≥ 1.

Application $\pi : X \to U$ sm. proper morphism of varieties / field k.

For each i, $\rho_i : \pi_1^{et}(U_{\bar{k}}, u) \to GL((R^i \pi_* \mathbb{Q}_\ell)_u)$

Thm (Deligne) ρ_i are semisimple.

Pf (i) π, X, U are all defined over some f.g. $\mathbb{Z}$-algebra. specialize to finite field.

(ii) Let $E \subseteq R^i\pi_*\mathbb{Q}_\ell$ be a lc subsheaf

Want: $0 \to E \to R^i\pi_*\mathbb{Q}_\ell \to F \to 0$ (*) to split.

$$(*) \in \mathrm{Ext}^1_{\pi_1^{et}(U_{\overline{\mathbb{F}_q}})}(F, E)^{\mathrm{Frob}}$$

(after replacing k w/ finite extn).

(iii)

$$\mathrm{Ext}^1_{\pi_1^{et}(U_{\overline{\mathbb{F}_q}})}(F, E)^{\mathrm{Frob}} = 0$$

$$\shortparallel$$

$$H^1(\pi_1^{et}(U_{\overline{\mathbb{F}_q}}), \underline{\mathrm{Hom}}(F, E))^{\mathrm{Frob}} =$$

$$H^1(U_{\overline{\mathbb{F}}_q, \text{ét}}, \underline{\text{Hom}}(F, E))^{\text{Frob}}$$

F, E have the same wt, b/c subquot of $R^i\pi_* \mathbb{Q}_\ell$ (by Weil conj.)

$$\Rightarrow \underline{\text{Hom}}(F, E) \text{ has wt } 0$$

$\overset{\text{Weil II}}{\Rightarrow}$ $H^1(U_{\overline{\mathbb{F}}_q}, \underline{\text{Hom}}(F, E))$ is mixed of wts $\{1, 2\}$

$\Rightarrow$ 1 not an eigenvalue of Frob

$\Rightarrow H^1(U_{\overline{\mathbb{F}}_q}, \underline{\text{Hom}}(F, E))^{\text{Frob}} = 0.$

Application 2: Chebotarev $\quad \rho : \pi_1^{\text{ét}}(X) \to G$

__Thm__ (Serre) X normal, $f : Y \to X$ G-cover, X, Y geom-conn'd $\mathbb{F}_q$-varieties. Let $C \subseteq G$ be a conjugacy class.

Then $\dfrac{\#\{x \in X(\mathbb{F}_{q^n}) \mid \rho(\mathrm{Frob}_x) \in C\}}{\# X(\mathbb{F}_{q^n})} \to \dfrac{|C|}{|G|}$

as $n \to \infty$. (Will see explicit error terms)

Pf Let $\mathbb{1}_C : G \to \mathbb{Q}_\ell$ be the indicator function of C.

Want to count

$$\sum_{x \in X(\mathbb{F}_{q^n})} \mathbb{1}_C(\rho(\mathrm{Frob}_x)) =$$

$$\sum_{x \in X(\mathbb{F}_{q^n})} \sum_{\chi_i \in Ch(G)} a_i \chi_i(\rho(\mathrm{Frob}_x))$$

$$= \frac{1}{|G|} \sum_{x \in X(\mathbb{F}_{q^n})} \sum_{\chi_i \in Ch(G)} |C| \chi_i([C]) \chi_i(\rho(\mathrm{Frob}_x))$$

$$= \frac{|C|}{|G|} \sum_{x \in X(\mathbb{F}_{q^n})} \sum_{\chi_i \in Ch(G)} \chi_i([C]) \chi_i(\rho(\mathrm{Frob}_x))$$

$$\pi_1^{et}(X) \xrightarrow{\rho} G \xrightarrow{\rho_{\chi_i}} GL_{n_i}(\overline{\mathbb{Q}}_\ell)$$

$$\rightsquigarrow \widetilde{\mathcal{F}}_{\chi_i}$$

$$\chi_i(\rho(\mathrm{Frob}_x)) = \mathrm{Tr}(\mathrm{Frob}_x \mid (\widetilde{\mathcal{F}}_{\chi_i})_x)$$

Sum becomes:

$$\frac{|C|}{|G|} \sum_{\chi_i \in Ch(G)} \sum_{x \in X(\mathbb{F}_{q^n})} \chi_i(C)\, \mathrm{Tr}(\mathrm{Frob}_x \mid (\widetilde{\mathcal{F}}_{\chi_i})_x)$$

$$= \frac{|C|}{|G|} \sum_{\chi_i \in Ch(G)} \chi_i(C) \sum_{i=0}^{2\dim X} (-1)^i\, \mathrm{Tr}(\mathrm{Frob}^n \mid H^i_c(X_{\overline{\mathbb{F}}_q}, \widetilde{\mathcal{F}}_{\chi_i}))$$

(i) χ_i - trivial

$$\sum_{i=0}^{2\dim X} \mathrm{Tr}(\mathrm{Frob}^n \mid H^i_c(X_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)) = \# X(\mathbb{F}_{q^n})$$

(ii) χ_i - non-trivial, Want

$$\sum_{i=0}^{2\dim X} \mathrm{Tr}(\mathrm{Frob}^n \mid H^i_c(X_{\overline{\mathbb{F}}_q}, \mathcal{F}_{\chi_i}))$$

small.

Small: $T(n, X)$

Want: $\frac{T(n, X)}{\# X(\mathbb{F}_{q^n})} \to 0$ as $n \to \infty$.

$\# X(\mathbb{F}_{q^n})$

$H_c^{2\dim X}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_{X_i})$ dual to $H^0(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_{X_i}^\vee(\dim))$

$= 0$ b/c X_i are non-trivial irreducibles.

Get contributions to $T(n, X)$ from

$H_c^i(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_{X_i})$ for $i < 2\dim X_i$

$\Rightarrow$ eigenvalues of Frob have abs. value

$q^{ni/2}$ for $i < 2\dim X$.

$\Rightarrow T(n, X) \leq q^{\frac{2\dim X - 1}{2} \cdot n} \cdot \sum \dim H_c^i(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_{X_i})$

$\Rightarrow \dfrac{T(n, X)}{q^{\dim X \cdot n}} \to 0$ as $n \to \infty$.

$\nwarrow$ $\# X(\mathbb{F}_{q^n})$ grows like $q^{\dim X \cdot n}$ ☐

Softer version

$\# X(\mathbb{F}_{q^n})$ grows like

$q^{n \cdot \dim X}$ (Lang-Weil)

Error term: $q^{n(\dim X - \frac{1}{2})}$ · Betti # 's