Étale Cohomology - 12/1/2020

Announcement: Thursday is last official class.

MAIN LEMMA

$X_0 \subseteq \mathbb{P}^1_{\mathbb{F}_q}$, $X = (X_0)_{\overline{\mathbb{F}}_q}$, $\mathcal{E}$ - lc $\mathbb{Q}_\ell$-sheaf on X_0

E - corresponding π_1-rep. Assume

(1) $F_x \curvearrowright \mathcal{E}_x$ rat'l char poly for $x \in |X|$

(2) non-deg skew-symmetric form

$\psi : E \times E \to \mathbb{Q}_\ell(-n)$

(3) $\pi_1(X) \to Sp(E, \psi)$ open image

Then: (a) E has wt n:

$F_x \curvearrowright E_x$ had eigenvalues α s.t.

$|\alpha| = q^{n(\deg x)/2}$

(b) $F \curvearrowright H^1_c(X, \mathcal{E})$ rat'l char poly,

eigenvalues β satisfy $|\beta| \leq q^{n/2+1}$

(c) $F \curvearrowright H^*(\mathbb{P}^1, j_* \mathcal{E})$ has rat'l char poly,

$j : X \hookrightarrow \mathbb{P}^1$

eigenvalues γ satisfy $q^{n/2} \leq |\gamma| \leq q^{n/2+1}$.

Last time:

reduced (a) to

<u>Lemma 1</u> $(\bigotimes_{2k} E)_{\pi_1(X)} \cong \bigoplus \mathbb{Q}_\ell(-kn)$

<u>Lemma 2</u> If $\forall k$, $\zeta(X, E^{\otimes 2k}, t)$ converges for $t < \frac{1}{q^{kn+1}}$, E has wt n.

<u>Pf of Lemma 1</u>

$k=1$: $(E \otimes E)_{\pi_1(X)}$

$\mathrm{Hom}_{\pi_1(X)}(E \otimes E, \mathbb{Q}_\ell) = \mathrm{Hom}_{\pi_1(X)}((E \otimes E)_{\pi_1(X)}, \mathbb{Q}_\ell)$

$= \mathrm{Hom}((E \otimes E)_{\pi_1(X)}, \mathbb{Q}_\ell)$

b/c π_1 has dense image in $Sp(E, \psi)$ $\quad = \mathrm{Span}(\psi)$

so $(E \otimes E)_{\pi_1(X)} = (E \otimes E)_{Sp(E,\psi)}$

<u>k general</u>: $\mathrm{Hom}((E^{\otimes 2k})_{Sp(E,\psi)}, \mathbb{Q}_\ell) =$

$\mathrm{Span}(\prod \psi(v_{i_1}, v_{i_2})) =$

$\mathbb{Q}_\ell(-kn)^{\oplus N}$

b/c each ψ contributes $\mathbb{Q}_\ell(-n)$.

Pf of Lemma 2:

$$S(X_0, \mathcal{E}^{\otimes 2k}, t) = \prod_{x \in |X|} \det(1 - F_x t^{\deg x} \mid \mathcal{E}_x^{\otimes 2k})^{-1}$$

Hypothesis $\Rightarrow$ $\dfrac{1}{\det(1 - F_x t^{\deg x} \mid \mathcal{E}_x^{\otimes 2k})}$ converges

for $|t| < \dfrac{1}{q^{kn+1}}$

$\Rightarrow$ each eigenvalue of Frobenius satisfy

$$|\alpha| > q^{(\deg x)\,kn+1}$$

$\xRightarrow{\text{tensor product trick}}$ $|\alpha| \geq q^{(\deg x)\,n/2}$

$\xRightarrow{\text{pairing}}$ $|\alpha| \leq q^{(\deg x)\,n/2}$. $\square$

Pf of (b) $S(X_0, \mathcal{E}, t)$ - controlled by $H^i_c(X, \mathcal{E})$

$$H^0_c(X, \mathcal{E}) = 0$$

$$H^2_c(X, \mathcal{E}) = E_{\pi_1(X)}(-1) = E_{Sp(E,\psi)}(-1)$$
$$= 0$$

$$\zeta(X_0, \mathcal{E}, t) = \det(1 - F^* t \mid H^1_c(X, \mathcal{E}))$$

$$= \prod_{x \in |X|} \frac{1}{\det(1 - F_x t^{\deg x} \mid \mathcal{E}_x)}$$

rat'l coeffs

$\Rightarrow \det(1 - F^* t \mid H^1_c(X, \mathcal{E}))$ has rat'l coeffs.

To understand eigenvalues, suffices to show

$$\prod \det(1 - F_x t^{\deg x} \mid \mathcal{E}_x) \text{ converges } |t| < \frac{1}{q^{n/2+1}}$$

Let $a_{i,x}$ be eigenvalues of $F_x \curvearrowright \mathcal{E}_x$

ETS $\displaystyle\sum_{i,x} |\alpha_{i,x} t^{\deg x}|$ converges for $|t| < \frac{1}{q^{n/2+1}}$

Follows from: (1) $|\alpha_{i,x}| = q^{(\deg x) n/2}$ (by (a))

(2) # of closed pts of deg $n \le q^{n+1}$

$\blacksquare$

Pf of (c)

$$0 \to j_! \mathcal{E} \to j_* \mathcal{E} \to i_* i^* j_* \mathcal{E} \to 0$$

skyscraper sheaf

$$X_0 \xhookrightarrow{j} \mathbb{P}^1 \xhookleftarrow{i} \mathbb{P}^1 \setminus X_0$$

LES in coh:

$$0 \to H^0(\mathbb{P}^1, j_* \mathcal{E}) \to H^0(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, i_* i^* j_* \mathcal{E}) \xrightarrow{\delta} \underset{\substack{\shortparallel \\ H^1(\mathbb{P}^1, j_! \mathcal{E})}}{H^1_c(X, \mathcal{E})} \to H^1(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, j_* \mathcal{E}) \to 0$$

$F \curvearrowright H^1(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, j_* \mathcal{E})$ has rat'l char poly:

$$\operatorname{ch}(F \mid H^1(\mathbb{P}^1_{\overline{\mathbb{F}}_q}, j_* \mathcal{E})) = \frac{\operatorname{ch}(F \mid H^1_c(X, \mathcal{E}))\, \operatorname{ch}(H^0(\mathbb{P}^1, j_* \mathcal{E}))}{\operatorname{ch}(F \mid \operatorname{im} \delta)}$$

Want: $q^{w/2} \le |\alpha| \le q^{w/2+1}$ immediate from (b).

Other inequality: PD gives perfect pairing

$$H^1(\mathbb{P}^1, j_* \mathcal{E}) \times H^1(\mathbb{P}^1, j_* \check{\mathcal{E}}(1)) \to H^2(\mathbb{P}^1, \mathbb{Q}_\ell(1))$$

applying upper bd to $\check{\mathcal{E}}(1)$ for the result.

$\boxtimes$

Understand $j_* \mathcal{E}$ for $\mathcal{E}$ lc on X:

<u>Prop</u> $U \subseteq Y$ of the open inside sm. proper geom conn'd curve $/k = \bar{k}$. $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ or $\mathbb{Q}_\ell$, $\ell \neq \mathrm{char}(k)$

$\tilde{\mathcal{F}}$ sheaf of Λ-modules on Y.

(a) $\bar{\mathcal{F}} \xrightarrow{\sim} j_* j^* \bar{\mathcal{F}}$ isom. iff

(i) $\forall s \in Y \setminus U$, cospecialization map

$\hat{\mathcal{F}}_s \to \bar{\mathcal{F}}_{\bar{\eta}}$ is injective

(ii) image is $\tilde{\mathcal{F}}_{\bar{\eta}}^{I_s}$

(b) For lc $\mathbb{Q}_\ell$-sheaf satisfying the above and locally constant on U

$H^r(Y, j_* \bar{\mathcal{F}}) \times H^{2-r}(Y, j_* \bar{\mathcal{F}}^\vee(1)) \to$

$H^2(Y, \mathbb{Q}_\ell(1))$

perfect pairing.

Pf (a) local computation
(b) Verdier duality.

Cohomology of Lefschetz Pencils:

X - sm. proj. variety, $X \hookrightarrow \mathbb{P}^n$

$L \subseteq \check{\mathbb{P}}^n$ Lefschetz pencil

$$\begin{array}{ccc} \mathrm{Bl}_{\text{base locus } L \cap X} X & = & X_+ \\ & & \downarrow \pi \\ & & L = \mathbb{P}^1 \end{array}$$

family s.t. gen. fiber is smooth, singular fibers have unique singularities which are ordinary double pts

$S \subseteq \mathbb{P}^1$:= set of $t \in \mathbb{P}^1$ s.t. X_t is singular.

Assume fiber dim'n $n = 2m+1$ odd.

$U = \mathbb{P}^1 \setminus S$ $\quad I_s$ = tame inertia at s

$V = (R^n \pi_* \mathbb{Q}_\ell)_{\bar{\eta}}$

<u>Claim</u>

(a) For $r \neq n, n+1$ $R^r \pi_* \mathbb{Q}_\ell$ is locally constant (hence constant).

(b) $R^n \pi_* \mathbb{Q}_\ell |_U$ loc. constant + <u>tame</u>.

(c) For each $s \in S$, $\exists$ "vanishing cycle"

$\delta_s \in V(m)$ well-defined up to sign

Span(δ_s) dual to coker of cospecialization map

$V_s \longleftarrow V_{\bar{\eta}}$

(d) $0 \to H^n(X_s, \mathbb{Q}_\ell) \xrightarrow{\text{cosp}} H^n(X_{\bar{\eta}}, \mathbb{Q}_\ell) \xrightarrow{\cdot \cup \delta_s} \mathbb{Q}_\ell(m-n) \to 0$

exact

(e) $\sigma_s \in I_s = \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \xrightarrow{t_\ell} \mathbb{Z}_\ell(1)$

Given $x \in V$

$$\sigma_s(x) = x \pm t_e(\sigma_c)(x \vee \delta_s)\,\delta_s$$