Brauer Groups

Reminder:

Goal: C sm. curve$/k=\bar{k}$, $H^i(C, \mathbb{G}_m) = \begin{cases} \mathcal{O}_C^\times(C) & i=0 \quad ✓ \\ \mathrm{Pic}(C) & i=1 \quad ✓ \\ 0 & i>1 \quad ? \end{cases}$

Today: $i=2$ — reduced this to understanding

$H^i(k(C), \mathbb{G}_m) \leftarrow ?$

$H^i(K_{\bar{x}}, \mathbb{G}_m) \leftarrow ?$

$\uparrow$ strict henselization of $\mathcal{O}_{C,x}$.

Defn X-scheme

$$\mathrm{Br}^{coh}(X) = \mathrm{Br}'(X) := H^i(X_{ét}, \mathbb{G}_m)_{tors}.$$

$$\bigcup_n \{ \overset{\text{étale-locally trivial}}{PGL_n\text{-torsors}} / X \} \xrightarrow{\delta} H^2(X_{ét}, \mathbb{G}_m)$$

$\mathrm{Br}(X) :=$ image of this map δ

Defn (δ) Bdry map

$$\bigcup_n H^1(X_{ét}, PGL_n) \xrightarrow{\delta} H^2(X_{ét}, \mathbb{G}_m)$$

arising from SES (Reference: Giraud, "Cohomologie Non-Abélienne")

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

of sheaves of gps on $X_{ét}$.

Rem "Long exact sequence of sets" terminates at $H^2(X_{ét}, \mathbb{G}_m)$.

Make δ explicit in terms of Čech cohomology:
$[T] \in H^1(X_{et}, PGL_n)$ T - PGL_n-torsor
split by some $\mathcal{U} \to X$.

On $\mathcal{U} \times_X \mathcal{U}$, descent data is given by some $\Gamma(\mathcal{U} \times_X \mathcal{U}, PGL_n)$
satisfies the cocycle condition.

(After refining $\mathcal{U}$) Lift this descent data to

$$s \in \Gamma(\mathcal{U} \times_X \mathcal{U}, GL_n)$$

$$\pi_{23}^* s \cdot \pi_{12}^* s \cdot (\pi_{13}^* s)^{-1} \in \Gamma(\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U}, \mathbb{G}_m)$$

2-cocycle representing an elt of
$H^2(X_{et}, \mathbb{G}_m)$. (exercise).

Slogan: $\delta([T])$ - obstruction to lifting T to a GL_n-torsor.
Brauer class of T.

Geometric interpretations of PGL_n-torsors + Brauer classes:

General principle: $T \in Sh^{sets}(X_{et})$ (think T is a scheme).
$$G = \underline{Aut}(T)$$

$$\{\text{locally split } G\text{-torsors}\} \xleftrightarrow{\sim} \{\text{forms of } T\}$$
sheaf on X_{et}
locally iso. to T.

$$\underline{Isom}(F, T) \leftarrow\!\!\!\sim\!\!\sim\!\!\sim F$$

$$\tau \rightsquigarrow (\tau \times T)/G \quad \leftarrow \text{sheaf quotient}$$

Ex $\{GL_n\text{-torsors}\} \leftrightarrow \{\text{v.b.'s}\}$

$G = PGL_n$: Objects w/ $\mathrm{Aut} = PGL_n$

Ex $\mathrm{Aut}_X(\mathbb{P}^{n-1}_X) = PGL_n$

(exercise)

Cor $\{PGL_n\text{-torsors}\} \leftrightarrow \{\text{forms of } \mathbb{P}^{n-1}\}$

Severi-Brauer X-schemes

Ex $\mathrm{Mat}_{n\times n}(A)$ has $\mathrm{Aut} = PGL_n$

(Noether-Skolem Theorem)

$\{PGL_n\text{-torsors}\} \leftrightarrow \{\text{forms of } \mathrm{Mat}_{n\times n}\}$

$\mathrm{End}_{\mathcal{O}_X}(\mathcal{O}_X^n)$

Azumaya algebras.

Twisted Sheaves

$U \to X$ étale cover, $\alpha \in \Gamma(U \times_X U \times_X U, \mathbb{G}_m)$

α represents $[\alpha] \in H^2(X_{\acute{e}t}, \mathbb{G}_m)$

An α-twisted sheaf is a qcoh sheaf $\mathcal{F}$ on U

an isom $\varphi : \pi_1^* \mathcal{F} \xrightarrow{\sim} \pi_2^* \mathcal{F}$

which satisfies the cocycle condition up to α.

$$\pi_{23}^* \varphi \circ \pi_{12}^* \varphi = \alpha \cdot \pi_{13}^* \varphi.$$

$QCoh(X,\alpha)$ - objects are α-twisted sheaves
- morphisms are morphisms of sheaves on $\mathcal{U}$ commuting w/ φ.

<u>Ex</u> $QCoh(X,1) \simeq QCoh(X)$. (étale descent).

<u>Prop</u> α, α' are 2-cocycles for $\mathbb{G}_m$

(1) $[\alpha] \in Br(X) \Leftrightarrow \exists$ α-twisted vector bundle.

(2) $QCoh(X,\alpha)$ is an Abelian category w/ enough injectives (if X is nice)

(3) $\otimes : QCoh(X,\alpha) \times QCoh(X,\alpha') \to QCoh(X, \alpha\cdot\alpha')$

$Hom : QCoh(X,\alpha) \times QCoh(X,\alpha') \to QCoh(X, \alpha^{-1}\alpha')$

(4) $Sym^n, \Lambda^n : QCoh(X,\alpha) \to QCoh(X, n\alpha)$

(5) $QCoh(X,1) \xrightarrow{\sim} QCoh(X)$.

<u>Pf</u> Exercise: Try (3,4).

<u>Cor</u> $Br(X)$ is a gp.

<u>Pf</u> α, α' $\mathcal{E}$: α-twisted v.b. $\mathcal{E}'$: α'-twisted v.b.

- $\alpha\cdot\alpha'$ — $\mathcal{E}\otimes\mathcal{E}'$ is $\alpha\alpha'$-twisted v.b.

- α - Brauer class, $\mathcal{E}$ α-twisted v.b.

Want: α^{-1}-twisted v.b. $\mathcal{E}^\vee$. $\square$

Prop α 2-cocycle for $\mathbb{G}_m$

$[\alpha]$ trivial $\Leftrightarrow$ $\exists$ α-twisted line bundle.

Pf $\Longrightarrow$: $\mathcal{O}_X$

Lemma If $[\alpha] = [\alpha']$, $QCoh(X, \alpha) \simeq QCoh(X, \alpha')$

(easy exercise)

$\Longleftarrow$: Descent data for $\mathcal{L}$ $\rightsquigarrow \beta \in \Gamma(U \times_X U, \mathbb{G}_m)$

$\check{\delta}(\beta) = \alpha$. $\square$

Cor Suppose $\mathcal{E}$ is an α-twisted v.b. of rk n.

Then $[\alpha] \in H^2(X_{et}, \mathbb{G}_m)$ is n-torsion.

Pf $\bigwedge^n \mathcal{E}$ is an α^n-twisted line bundle $\Rightarrow$

$[\alpha^n] = [\alpha]^n$ is trivial.

Cor $Br(X) \subseteq Br'(X) := H^2(X_{et}, \mathbb{G}_m)_{tors}$.

Examples of Brauer classes

(1) $\underset{X}{\{x^2+y^2+z^2=0\}}$ $/\mathbb{R}$ - sm. conic w/ no rat'l pts

$X_{\mathbb{C}} \simeq \mathbb{P}^1 \Rightarrow X$ twisted form of $\mathbb{P}^1_{\mathbb{R}}$.

$\delta([X])$ generates $Br(\mathbb{R}) = \mathbb{Z}/2$.

(2) Hamilton Quaternions are a division algebra (hence Azumaya algebra) representing the same elt.

$$\begin{tikzcd}
 & \text{Severi-Brauer} \\
\alpha\text{-Twisted sheaf } \mathcal{E} \arrow[ur, "\mathbb{P}(-)"] \arrow[r] & \mathrm{End}(\mathcal{E}) \text{ Azumaya algebra} \arrow[u, dashed, "\text{moduli of certain ideals in Azumaya}"']
\end{tikzcd}$$

(3) $Br(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$

Rem / k a field, any 2-torsion Brauer class is rep'd by "a quaternion algebra"

(4) $$0 \to Br(\mathbb{Q}) \to \bigoplus_{\substack{v \text{ place} \\ \text{of } \mathbb{Q}}} Br(\mathbb{Q}_v) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \to 0$$

Exercise Use the SB interpretation to show that if $\alpha \in Br(\mathbb{Q})$, $\alpha|_{\mathbb{Q}_v} = 0$ for almost all v.

Interpret multiplication:

A_1, A_2 are Azumaya algebras rep'ing α_1, α_2.

$A_1 \otimes A_2$ is Azumaya alg. rep'ing $\alpha_1 \alpha_2$.

$P_1 = \mathbb{P}(\mathcal{E})$, $P_2 = \mathbb{P}(\mathcal{E}')$ are SB's rep'n α_1, α_2

($\mathcal{E}$: α_1-twisted v.b., $\mathcal{E}'$: α_2-twisted v.b.)

$\mathbb{P}(\mathcal{E} \otimes \mathcal{E}')$ reps $\alpha_1 \alpha_2$.

Q (Period-index question) Given $\alpha \in Br(X)$

What is the minimum rk, gcd of ranks... of an α-twisted v.b.

Next time: Understand $H^i(K, \mathbb{G}_m)$

Thm (1) $k(C)$ fn. field of a curve $/ k = \bar{k}$

$$H^2(k(C), \mathbb{G}_m) = 0$$

(2) $K_{\bar{x}}$ strictly Henselian dvr

$$H^2(K_{\bar{x}}, \mathbb{G}_m) = 0.$$