EC 10-6-2020

Last time: étale cohomology of curves

Today: compactly supported cohomology + Gysin

Extn by zero

$U \xhookrightarrow{j} X$ open embedding

Defn (extn by zero) $j_! : Sh^{ab}(U) \to Sh^{ab}(X)$

$$j_!(V) = \left(\begin{cases} \mathcal{F}(V \times_X U) & \text{if } im(V) \subseteq X \\ & \subseteq U \\ 0 & \text{otherwise} \end{cases} \right)^a$$

presheaf

Rem $j_!$ exists for more general morphisms but requires input from Nagata compactification.

Prop (exercise) $j_!$ is left adjoint to j^*

Pf Check this on presheaves + and use adjoint property of sheafification.

Prop $(j_! \mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & \bar{x} \in im\, j \\ 0 & \bar{x} \notin im\, j \end{cases}$

Cor $j_!$ is exact

Pf check on stalks $\boxtimes$

Useful for "excision":

Prop $\mathcal{F} \in Sh^{ab}(X_{et})$ $\quad U \underset{open}{\overset{j}{\hookrightarrow}} X \underset{cl}{\overset{\iota}{\hookleftarrow}} Z = X \setminus U$

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to \iota_* \iota^* \mathcal{F} \to 0$$

exact.

Pf Check on stalks. Choose $\bar{x} \in X$ geom. pt

Case 1: $\bar{x} \in U$

$$0 \to \mathcal{F}_{\bar{x}} \xrightarrow{id} \mathcal{F}_{\bar{x}} \to 0 \to 0$$

Case 2: $\bar{x} \in Z$

$$0 \to 0 \to \mathcal{F}_{\bar{x}} \xrightarrow{id} \mathcal{F}_{\bar{x}} \to 0 \quad \square$$

Defn (Cohomology w/ cpct support)

$\mathcal{F} \in Sh^{ab}(U_{et})$, $\quad j: U \hookrightarrow X$ open, X proper.

$$H^i_c(U_{et}, \mathcal{F}) := H^i(X_{et}, j_! \mathcal{F}).$$

Q 1) Why does X exist?

2) Why is this defn independent of j, X.

(1) Thm (Nagata)

$$\begin{tikzcd} & X \arrow[dr, "\text{proper}"] & \\ U \arrow[ur, "\exists\ \text{open}"] \arrow[rr, "\text{sep'd}"'] & & S \end{tikzcd}$$

(2) Independence for torsion sheaves will require proper base change.

<u>Prop/Ex.</u> U conn'd regular curve$/k=\bar{k}$

char $k \nmid n$.

Canonical iso $H^2_c(U_{et}, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}$.

<u>Pf</u>

$$\begin{tikzcd} U \arrow[r, hook, "j"] & X & Z = X \setminus U \arrow[l, "i"'] \end{tikzcd} \leftarrow \text{canonical regular compactification}$$

<u>Want</u>: $H^i(X, j_! \mu_n)$

$$0 \to j_! j^* \mu_n \to \mu_n \to i_* i^* \mu_n \to 0$$

$j_! j^* \mu_n = j_! \mu_n$

$i_* i^* \mu_n$: $\oplus$ skyscraper sheaves.

$$\cdots \longrightarrow H^i_c(U, \mu_n) \to H^i(X, \mu_n) \to H^i(X, i_* i^* \mu_n) \to H^{i+1}_c(U, \mu_n) \to \cdots$$

$$H^i(X, i_* i^* \mu_n) = \begin{cases} \bigoplus_{\#\text{pts}} \mu_n(k) & i=0 \\ 0 & i>0 \end{cases}$$

$$H^i(X, R^j i_* i^* \mu_n) \Rightarrow H^{i+j}(Z, \mu_n)$$

$R^j i_* i^* \mu_n = 0$ for $j > 0$.

b/c i_* is exact.

$$0 \to \underset{\substack{\| \\ H^0(X,\, j_! \mu_n) \\ \| \\ 0}}{H^0_c(U, \mu_n)} \to \underset{\substack{\| \\ \mu_n(k)}}{H^0(X, \mu_n)} \xrightarrow{\Delta} \bigoplus_{\# pts} \mu_n(k) \to H^1_c(U) \to \underset{\substack{\| \\ \mathrm{Pic}\, X[n]}}{H^1_c(X)}$$

$$\to 0 \to H^2_c(U, \mu_n) \to \underset{\substack{\| \\ \mathbb{Z}/n\mathbb{Z}}}{H^2(X, \mu_n)} \to 0$$

Proper Base Change + Finiteness results

Defn $\mathcal{F} \in Sh(X_{\acute{e}t})$ is constructible

if (a) For every $i : Z \xhookrightarrow{cl} X$

$\exists$ non-empty open $U \subseteq Z$ s.t.

$i^* \mathcal{F}|_U$ is locally constant.

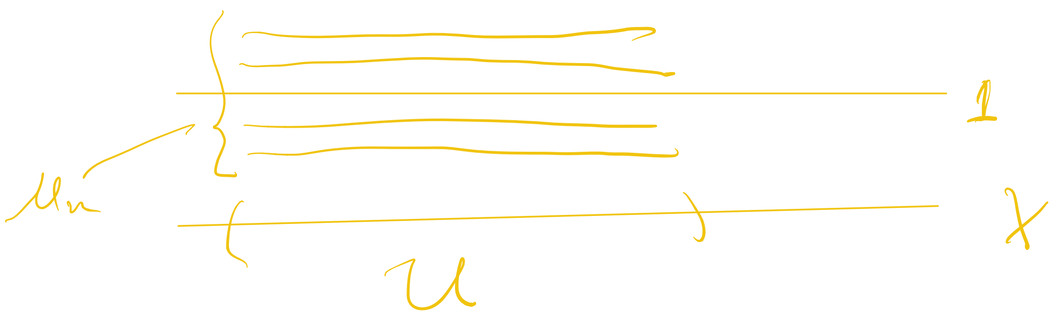
$\exists$ cover $V \to U$ s.t. $i^* \mathcal{F}|_V$ is constant.

(b) stalks of $\mathcal{F}$ are finite.

Ex $U \xrightarrow{\text{open}} X$, $j_!\mu_n$ is constructible.

Ex $\mathcal{F}$ rep'd by a quasi-finite X-scheme.

Picture of $j_!\mu_n$:



Thm (Hard) $f: X_{\acute{E}t} \to X_{\acute{e}t}$, $\mathcal{F} \in Sh(X_{\acute{E}t})$

s.t. (1) $\mathcal{F} \leftarrow f^* f_* \mathcal{F}$ isom.

(2) $f_* \mathcal{F}$ constructible

Then $\mathcal{F}$ is rep'd by a quasi-finite X-scheme.

Thm $\pi: X \to S$ proper morphism, $\mathcal{F} \in Sh^{ab}(X_{\acute{e}t})$

Then, $R^i \pi_* \mathcal{F}$ are constructible for $i \geq 0$

and $(R^i\pi_*\mathcal{F})_{\bar{s}} \xrightarrow{\sim} H^i(X_{\bar{s}}, \mathcal{F}|_{X_{\bar{s}}})$.

for all geom. pts $\bar{s} \in S$.

<u>Cor</u> X proper$/_{k=k^s}$, $\mathcal{F} \in Sh^{ab}(X_{et})$ constructible

(a) $H^i(X_{et}, \mathcal{F})$ is finite

(b) $k \subset L$ extn of seplly closed fields

$$H^i(X_{et}, \mathcal{F}) \xrightarrow{\sim} H^i(X_{L,et}, \mathcal{F}|_{X_{L,et}}).$$

<u>Pf</u> (a) constructibility

(b) $(R^i\pi_*\mathcal{F})_{\mathrm{Spec}\, L \to \mathrm{Spec}\, k}$

<u>Non-ex</u> $X = \mathbb{A}^1_{\mathbb{F}_p}$, $\mathcal{F} = \mathbb{Z}/p\mathbb{Z}$

$H^1(X_{et}, \mathbb{Z}/p\mathbb{Z})$ infinite.

<u>Cor</u> (Proper base change theorem)

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

For any $\mathcal{F} \in Sh^{ab}(X_{et})$

$f^*(R^i\pi_*\mathcal{F}) \to R^i\pi'_* f'^*\mathcal{F}$

If π proper, $\mathcal{F}$ torsion, this map is isom.

<u>Pf</u> Construct this map using adjointness

Check on stalks (for constr. sheaves) $\square$