Étale Cohomology

Brauer gps + H^i(curves)

Last time: Brauer gps, α - 2-cocycle rep'n a class $[\alpha] \in Br(X)$

twisted v.b. $\mathcal{E} \mapsto \mathbb{P}(\mathcal{E})$ severi-Brauer schemes

$\{\alpha\text{-twisted sheaves}\} \to$ severi-Brauer schemes

$\{\alpha\text{-twisted sheaves}\} \to$ Azumaya algebras

twisted v.b. $\mathcal{E} \mapsto \mathcal{E}nd(\mathcal{E}) \leftarrow$ coh. sheaf of algebras

Thm (Tsen's theorem) Suppose k is a C_1 field (quasi-alg. closed). Then $Br(k) = 0$.

k quasi-alg. closed if for any homog. poly $f \in k[x_1, \dots, x_n]$ w/ $\deg f < n$ has a non-trivial zero.

Thm K either
(1) function field of a curve/alg. closed field
(2) fraction field of strictly Henselian dvr

Then K quasi-alg. closed.

Cor $Br(K)$ (K as above) is trivial.

Defn (reduced norm) $\mathcal{E}$ - α-twisted sheaf

$\mathcal{E}nd(\mathcal{E})$ - Azumaya algebra

$$Nm: \mathcal{E}nd(\mathcal{E}) \to \mathcal{E}nd(\Lambda^{top}\mathcal{E}) = \mathcal{O}_X$$

is given by functoriality of Λ.

Prop Given $f \in \mathcal{E}nd(\mathcal{E})$, f is invertible if and only if $Nm(f)$ is a unit.

Pf Check locally.

Pf of Tsen's theorem (k q-alg. closed $\Rightarrow Br(k) = 0$)

Want: Given $[\alpha] \in Br(k)$, we can find an α-twisted line bundle.

Have: α-twisted v.b. $\mathcal{E}$.

Idea: Find a non-trivial sub-bdle of $\mathcal{E}$ if $rk\,\mathcal{E} > 1$.

(i) Find $f \in \mathcal{E}nd(\mathcal{E})$ which is not invtble, i.e. $Nm(f) = 0$.

$$\underset{(rk(\mathcal{E}))^2 \text{ dim'l affine space}}{Nm: \mathcal{E}nd(\mathcal{E})} \to k$$

Nm - polynomial ftn in $rk(\mathcal{E})^2$ ..

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$$\deg(\mathrm{Nm}) = \mathrm{rk}(\mathcal{E})$$

(ii) q. alg. closed $\Rightarrow$ Nm has a non-trivial zero if $\mathrm{rk}(\mathcal{E}) > 1$.

$\exists f \in \mathrm{End}(\mathcal{E})$ non-zero s.t. $\mathrm{Nm}(f) = 0$.

Set $\mathcal{E}' = \ker(f)$

α-twisted v.b. of $\mathrm{rk} < \mathrm{rk}\,\mathcal{E}$.

Keep going until $\mathrm{rk}\,\mathcal{E}' = 1 \Rightarrow [\alpha] = 0 \in \mathrm{Br}$. $\boxed{\Xi}$

Cor k q-alg. closed field, $H^2(k, \mathbb{G}_m) = 0$.

Pf For a field $H^2(k, \mathbb{G}_m) = \mathrm{Br}(k)$.

(write down explicit CSA using a cocycle)

Rem True in a lot of situation. Serre's Local fields.

Pf ($k = k(C)$ - ftn field of a curve/alg. closed field q. alg. closed)

Given $f \in k(C)[x_1, \ldots, x_n]$ homogeneous, s.t.

$$\deg f < n,$$

want a non-trivial zero of f in $k(C)^n$.

Idea: Choose an ample divisor D on C

$$
\begin{array}{ccc}
\sim m\cdot n & X: & \Gamma(C, \mathcal{O}(mD))^n \quad m \in \mathbb{Z} \\
\vee & & \downarrow f \\
\sim (\deg f) m & Y: & \Gamma(C, \mathcal{O}((\deg f) m D + D'))
\end{array}
$$

D' ← poles of coeffs.

map of affine spaces $/k$.

For $m \gg 0$, map from $X \to Y$, w/ $\dim X > \dim Y$

$\Rightarrow$ dim'n of any non-empty fiber > 0.

Want $\dim f^{-1}(0) > 0$

ETS: $f^{-1}(0)$ is non-empty.

$0 \in f^{-1}(0)$ ∎.

<u>Cor</u> $Br(k(C)) = H^2(k(C), \mathbb{G}_m) = 0$.

<u>Pf</u> (K - fraction field of a strictly Henselian dvr) is q.alg. closed.

Lang's thesis. <u>Exercise</u> Prove this when K is equicharacteristic zero. ∎

Cor $Br(K) = H^2(K, \mathbb{G}_m) = 0$.
for K fraction field of strictly Henselian dvr.

Thm K as above ($k(C)$ or $K_{\bar{x}}$)

$k(C)$: fn field of a curve/alg. closed field

$K_{\bar{x}}$: fraction field of strictly Henselian dvr.

Then $H^i(K, \mathbb{G}_m) = 0$ for all $i > 0$. (Tate's theorem)

Pf Goal: $H^i(L/K, \mathbb{G}_m) = 0$ for all finite sepble Galois extns L/K.

$$= H^i(\mathrm{Gal}(L/K), L^\times)$$

(1) We know this vanishing for $i = 1, 2$.
Hilbert 90 + Tsen's theorem + inflation-restriction.

(2) (L/K cyclic)

cohomology is 2-periodic!
and we know answer for cyclic gps.

(3) (L/K nilpotent)

$C \subseteq \mathrm{Gal}(L/K)$ normal + cyclic.

$$1 \to C \to \mathrm{Gal}(L/K) \to \underbrace{G'}_{\text{nilpotent}} \to 1$$

Inflation-restriction $\Rightarrow$ success!

(4) (L/k general)

p-groups are nilpotent $\Rightarrow$

For $G_p \subseteq \mathrm{Gal}(L/k)$ a p-sylow,

$H^i(G_p, L^\times) = 0$ (by part (3))

$$H^i(\mathrm{Gal}(L/k), L^\times) \hookrightarrow \bigoplus_p H^i(G_p, L^\times)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{0}$$

injective b/c

$$H^i(\mathrm{Gal}(L/k), L^\times) \underset{\text{cor.}}{\overset{\text{res}}{\rightleftarrows}} H^i(G_p, L^\times)$$

$\mathrm{cor} \circ \mathrm{res} = [G : G_p]$ ← prime to p.

$\Rightarrow$ res is injective away from prime-to-p-torsion.

$\boxed{\S}$

Cor C sm. curve/ alg. closed field.

$H^i(C_{\acute{e}t}, \mathbb{G}_m) = H^i(k(C), \mathbb{G}_m)$ for $i \geq 1$

Pf Before, $H^i(C_{\acute{e}t}, \mathbb{G}_m) \simeq H^i(C_{\acute{e}t}, \eta_* \mathbb{G}_m)$ for $i \geq 1$. (Divisor exact seq.)

Claim $H^i(C_{\acute{e}t}, \eta_* \mathbb{G}_m) = H^i(k(C), \mathbb{G}_m)$

Pf Leray spectral sequence ⇒ ets that

$R^j \eta_* \mathbb{G}_m = 0$ for $j > 0$.

$\subset$ stalks are

$H^i(X_{\bar{x}}, \mathbb{G}_m) = 0$.

$\boxed{S}$

Cor $H^i(C_{et}, \mathbb{G}_m) = H^i(k(C), \mathbb{G}_m) = 0$. $i > 1$

Pf Combine cor w/ Tate's theorem.

Cor $H^i(C_{et}, \mathbb{G}_m) = \begin{cases} \mathbb{G}_m(C) & i = 0 \\ \mathrm{Pic}(C) & i = 1 \\ 0 & i > 1 \end{cases}$

Cor $H^i(C_{et}, \mu_n)$ (n prime to the characteristic of k).

$$= \begin{cases} \mu_n & \text{if } i = 0 \\ \mathrm{Pic}(C)[n] & i = 1 \\ \begin{cases} \mathbb{Z}/n\mathbb{Z} & i = 2, \text{ if } C \text{ proper} \\ 0 & \text{otherwise} \end{cases} \\ 0 & i > 2. \end{cases}$$

Pf Kummer seq $1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$

Rem This is a Galois eqvt description if C proper.

Non-Galois equivariant description: $\mathrm{Pic}\, C[n] = (\mathbb{Z}/n\mathbb{Z})^{2g}$.

Rem ($/\mathbb{C}$) Surface of genus g,

coh w/ coeffs in $\mathbb{Z}/n\mathbb{Z}$ looks the same.

Next time: Building towards, Poincaré duality,

coh. w/ cpct support, ...